Physics 89 [illegible]

Taylor Series:

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \approx \frac{x^n}{n!}$

$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} \cdots \approx \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} \cdots \approx \frac{(-1)^n x^{2n}}{(2n)!}$

$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 \cdots \approx \binom{p}{n} x^n$, $\binom{p}{n} = \frac{p!}{n!(p-n)!}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$(e^{i\theta})^* = e^{-i\theta}$

Linearity:

$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(v_2)$

$f(a\vec{v}_1) = a f(\vec{v}_1)$

Vector Space V is closed under addition & scalar multiplication and contains $\vec{0}$

Subspace must contain $\vec{0}$

dummy indices summed over, may be replaced w/ an unused index

free index not summed over

$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$\vec{a} \cdot \vec{b} = \delta_{ij} \vec{a}_i \vec{b}_j$ $|a||b|\cos\theta$

$\epsilon_{ijk} \begin{cases} 1 & (ijk) = (1,2,3),(2,3,1),(3,1,2) \\ -1 & (ijk) = (3,2,1),(2,1,3),(1,3,2) \\ 0 & \text{else} \end{cases}$

$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{kji} = -\epsilon_{jik} = -\epsilon_{ikj}$

$(\vec{a} \times \vec{b})_i = \epsilon_{ijk} \vec{a}_j \vec{b}_k$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

$\epsilon_{jjk} = 0$

$\delta_{ij}\delta_{jk} = \delta_{ik}$

$\delta_{ij}\epsilon_{jkl} = \epsilon_{ikl}$

$\delta_{ij}\epsilon_{ijk} = \epsilon_{jjk} = 0$

$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$

$\epsilon_{ijk}\epsilon_{ijk} = 6$

basis: lin indep, span V

$C^i_j = A^i_k B^k_j$

$(A^T)^T = A$

$(AB)^T = B^T A^T$

$AB = 1 \Rightarrow B = A^{-1}$ unique if exists

$\vec{a} \cdot \vec{b} = 0$ if $\vec{a} \perp \vec{b}$

$\vec{a} \times \vec{b} = |a||b|\sin\theta$ RHR

$\vec{a} \times \vec{b} = 0$ if $\vec{a} \parallel \vec{b}$ or $\vec{a} \parallel -\vec{b}$

$\vec{a} \times \vec{a} = 0$

if $W = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} \neq 0 \Rightarrow f_1(x) \cdots f_n(x)$ are lin. indep.

basis: set of l.i. vectors that span a vector space

any vector is a linear comb. of basis

$\text{norm}(\vec{a}) = \sqrt{\vec{a} \cdot \vec{a}}$

Graham Schmidt Process

$w_1 = v_1$

$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$

$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$

$w_n = v_n - \sum_{i=1}^{n-1} \frac{v_n \cdot w_i}{w_i \cdot w_i} w_i$

Vector Space:
- closure
- vector addition
  - commutative
  - associative
  - $\vec{0}$ vector ($\vec{v} + \vec{0} = \vec{v}$)
  - additive inverse ($\vec{v} + -\vec{v} = \vec{0}$)
- scalar mult
  - distributive (sum of vectors & sum of scalars)
  - associative (prod. of scalars)
  - $0 \cdot \vec{v} = \vec{0}$ & $1 \cdot \vec{v} = \vec{v}$

Inner Prod on $a \le x \le b$

$\int_a^b A^*(x) B(x)\, dx$

$\langle A|B\rangle^* = \langle B|A\rangle$ $\langle A|A\rangle \ge 0$, $= 0$ iff $A = 0$

$\langle C|aA + bB\rangle = a\langle C|A\rangle + b\langle C|B\rangle$

$\langle aA + bB|C\rangle = a^*\langle A|C\rangle + b^*\langle B|C\rangle$

$\langle aA|bB\rangle = a^* b \langle A|B\rangle$

$(A^T)^i_j = A^j_i$

Row Operations:
- i Interchange 2 rows
- ii multiply by $c \in \mathbb{R}$ $c \neq 0$
- iii add multiples of rows together

rank = # of nonzero rows remaining after row reducing

rank(A) = rank(A^T)

M is augmented matrix except last col

A is augmented matrix

rank(M) < rank(A)

$0 \cdots 0 \mid x \in \mathbb{R}$

$\Rightarrow$ inconsistent/wrong/no solution

$0 \cdots 0 \mid 0 \Rightarrow$ infinite sols

rank(M) = rank(A) < # unknowns

rank(M) = rank(A) = # unknowns

$\Rightarrow$ unique sol

Determinants:

remember +/- $\begin{matrix} + & - & + & \cdots \\ - & & & \\ \vdots & & & \end{matrix}$

if a row/col is mult by k, det = k · det

det = 0 if a row/col = 0, 2 rows/cols are same, or 2 rows/cols are proportional

if 2 rows/cols change position, det changes sign

det(A) = det(A^T)

same if add to each row k · corresponding element of other row

can expand along row/col

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

order of largest nonzero det of square submatrix is rank

Cramer's Rule

D = det(M)

each var = $\frac{\text{val of det(M) if replace that var's col w/ coeff col}}{D}$

linear if $f(r_1 + r_2) = f(r_1) + f(r_2)$ & $f(ar) = af(r)$

matrix M of orthogonal transformation is orthogonal matrix $\Rightarrow M^{-1} = M^T$

$\Rightarrow \det(M) = \pm 1$

rotation

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(AB)^i_j = A^i_k B^k_j$

$\det(AB) = \det(BA) = \det(A)\det(B)$

$MM^{-1} = I$

$M^{-1} = \frac{1}{\det(M)} C^T$

C^i_j = cofactor of m_{ij} (det after removing i, j & multiply by $(-1)^{i+j}$)

Transpose $A^T, \tilde{A}, A', A^t$

Complex Conj $\bar{A}, A^*$

Hermitian $A^\dagger$

Inverse A^{-1}

Hermitian $\Rightarrow A = A^\dagger$

Symmetric $\Rightarrow A = A^T$

orthogonal $\Rightarrow A^{-1} = A^T$

complex conj of elements $(A^*)^T$

unitary $\Rightarrow A^{-1} = A^\dagger$

normal $\Rightarrow AA^\dagger = A^\dagger A$

$(ABC)^T = C^T B^T A^T$

$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

$A\vec{x} = \vec{0}$ $\vec{x}$ = nullspace

e-vals

$A\vec{v} = \lambda\vec{v}$

$\Rightarrow A\vec{v} = \lambda I\vec{v}$

$(A - \lambda I)\vec{v} = \vec{0}$

$\Rightarrow \det(A - \lambda I) = 0$

replace λ and row reduce $\left[A - \lambda I \middle| \begin{matrix}0\\0\\\vdots\\0\end{matrix}\right]$

for every λ

Diagonalizing

$C = [v_1, v_2 \dots v_n]$

for $\lambda_1 < \lambda_2 < \lambda_3 \dots < \lambda_n$

$D = C^{-1} M C$

↳ diagonal matrix
↳ similar to M

degeneracy - 2 or more e-$\vec{v}$'s correspond to the same e-val

Quadratic Form

$x^T M x$

M is symmetric

$ax^2 + 2bxy + cy^2$

$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$T^{\text{contravar}}_{\text{covar}}$

Tensors

type(p, r) tensor eats p dual vectors & r vectors to return a scalar

(p: lower indices, r: upper indices)

rank $p+r$

$\underline{\hat{e}}^i(\hat{e}_j) = \delta^i_j$

vectors transform w/ C^{-1} (contra)

$V||w| = v^i w^j g_{ij}$

$\alpha_{[f]j} = \alpha_{[e]i} C^i_j$, $C^i_j = (\hat{f}_i)^j_{[e]}$ ← new basis vec.

$\underline{\hat{f}}^i = \tilde{C}\underline{\hat{e}}^i = C^i_j \underline{\hat{e}}^j = (\underline{\hat{f}}^i)_{[e]k}$

$C\tilde{C} = I \Rightarrow$ change of ~~mat~~ basis for dual vectors is inv. of cob for vectors

contraction of type(p,q) w/ type(r,s)
$\rightarrow$ type$(p+r-1, q+s-1)$

tensor product (p,q) w/ (r,s)
$\rightarrow$ type$(p+r, q+s)$

factor of S^{-1} steals a lower index

factor of S steals an upper index

for S changes basis from $[e] \rightarrow [f]$

Jacobian

$J^i_j = \frac{\partial x^i}{\partial \tilde{x}_j}$ $(J^{-1})^i_j = \frac{\partial \tilde{x}_i}{\partial x^j}$

contravar transform w/ J^{-1}

covar w/ J

Cartesian Metric Tensor $g_{ij} = \delta_{ij}$

$\Rightarrow g_{i'j'} = J^i_{i'} J^j_{j'} g_{ij}$

Change of Var

↳ know derivatives using chain rule

Complementary Sol

characteristic eqn roots

$\lambda_1 \dots \lambda_n$

$y_c(t)$ $c_1 e^{\lambda_1 t} \dots c_n e^{\lambda_n t}$

if $\lambda_i = \lambda_j \Rightarrow c_i e^{\lambda_i t}$
$c_j t e^{\lambda_j t}$

$y = y_h + y_p$

fourier transform of derivative is $i\omega$ × fourier trans. of original fn

Differential Eqns

$\dot{y} + P(t)y = 0$

has sol. of form

$y(t) = y_0 e^{-I(t)}$

$I(t) = \int_0^t P(\hat{t})\,d\hat{t}$

$\dot{y} + P(t)y = Q(t)$

$y(t) = y_p(t) + y_h(t)$
$= y_p(t) + y_0 e^{-I(t)}$

$\dot{\vec{N}} = T\vec{N}$

$\Rightarrow \left(\frac{d}{dt} - T\right)\vec{N} = \vec{0}$

ansatz $y_h(t) = \vec{v} e^{\lambda t}$

for $P\vec{v} = \lambda\vec{v}$

$y_{hi}(t) = \vec{v}_i e^{\lambda_i t}$

$y_h(t) = \sum_i c_i \vec{v}_i e^{\lambda_i t}$

for $a_n \frac{d^n y}{dt^n} + \dots + a_1 \frac{dy}{dt} = 0$

$y = e^{\lambda t}$

Reduction of Order

knowing $y_1(x)$, let $y_2(x) = v(x) y_1(x)$

→ sub and solve for $v(x)$

change var to $u = v'$

Undetermined Coeffs

guess & check

if guess solves homogeneous ODE, multiply by powers of t

Table of Guesses

Inhomogeneity	Guess for $y_p(t)$	Undet. Coeffs	Misc
$ke^{\gamma t}$	$Ce^{\gamma t}$	C	
kt^n	$\sum_{p=0}^{n} c_p t^p$	$c_0 \dots c_n$	explicit lin. indep. of t^p terms
$\sum_{p=0}^{n} k_p t^n$	$\sum_{p=0}^{n} c_p t^p$	$c_0 \dots c_n$	same↑
$c_1 k\cos\omega t + c_2 k\sin\omega t$	$Ae^{i\omega t} + Be^{-i\omega t}$ $c_1\cos\omega t + c_2\sin\omega t$	c_1, c_2	lin indep of $\cos\omega t$ & $\sin\omega t$
$c_1 ke^{\alpha t}\cos\omega t + c_2 ke^{\alpha t}\sin\omega t$	$c_1 e^{\alpha t}\cos\omega t + c_2 e^{\alpha t}\sin\omega t$	c_1, c_2	lin indep of $e^{\alpha t}\cos\omega t$ & $e^{\alpha t}\sin\omega t$

$\int u\,dv = uv - \int v\,du$

Var of Parameters

$$\frac{d^2y}{dt^2} + B(t)\frac{dy}{dt} + C(t)y = f(t)$$

assuming y_h is known

$y_h(t) = c_1 y_{h_1}(t) + c_2 y_{h_2}(t)$

allowing c_1 & c_2 to be fns of t, $\Rightarrow y_p(t) = c_1(t)y_{h_1}(t) + c_2(t)y_{h_2}(t)$

$$\Rightarrow y_p(t) = -y_{h_1}(t)\int \frac{y_{h_2}(t)f(t)}{W(t)}dt + y_{h_2}(t)\int \frac{y_{h_1}(t)f(t)}{W(t)}dt$$

$W(t) = y_{h_1}\dot{y}_{h_2} - y_{h_2}\dot{y}_{h_1}$

Exact Differentials

of form $\frac{du}{dt} = N$, $\frac{du}{dy} = M$

$M(y,t)\dot{y} + N(y,t) = 0$ (1)

$$\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y} \quad (2)$$

if a form (1) does not satisfy (2), attempt integrating factor $\alpha(y,t)$: $e^{\int \frac{M_y - N_t}{N}dt}$ or $e^{\int \frac{N_t - M_y}{M}dy}$

$M(y,t)\alpha(y,t)\dot{y} + N(y,t)\alpha(y,t) = 0$

$$\frac{\partial(M\alpha)}{\partial t} = \frac{\partial(N\alpha)}{\partial y}$$

Euler/Cauchy Diff Eqns

$$a_2 t^2\frac{d^2y}{dt^2} + a_1 t\frac{dy}{dt} + a_0 y = f(t)$$

using $t = e^z$,

becomes $a_2\frac{dy^2}{dt^2} + (a_1 - a_2)\frac{dy}{dt} + a_0 y = f(e^z)$

Homogeneous under $t \to ct$ & $y \to cy$ is invariant $\Rightarrow y = tu$

No bare y

$P(t)\ddot{y} + Q(t)\dot{y}^2 = 0$

$v = \dot{y}$

No bare t

$m\ddot{y} = F(y)$

$v = \dot{y} \Rightarrow \ddot{y} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$

PDEs

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2}$$

$$D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

$B^2 - AC < 0 \Rightarrow$ elliptic PDE

$B^2 - AC = 0 \Rightarrow$ parabolic PDE

$B^2 - AC > 0 \Rightarrow$ hyperbolic PDE

Fourier Transform Soln

$f(x) = \int_{-\infty}^{\infty} c(k)e^{ikx}dk$ — spatial domain

$f(t) = \int_{-\infty}^{\infty} c(\omega)e^{i\omega t}d\omega$ — time domain

$c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx$ — wave-vector domain

$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt$ — freq. domain

General Method:

PDE $\xrightarrow[\text{trans}]{\text{fourier}}$ Simpler eqn. $\xrightarrow{\text{solve}}$ fourier trans. of soln $\xrightarrow[\text{fourier trans}]{\text{inv.}}$ soln

Separation of Variables

consider ansatz $u(x,y) = X(x)Y(y)$ for homogeneous PDE

separate into two ODEs, inhomogeneity is a constant

constants sum to 0

Boundary Conditions

fuck around

Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$\int_0^{2\pi} \sin(mt)\sin(nt)\,dt = \begin{matrix} 0 & m \neq n \\ \pi & m = n \end{matrix}$$

$$\int_0^{2\pi} \cos(mt)\cos(nt)\,dt = \begin{matrix} 0 & m \neq n \\ \pi & m = n \end{matrix}$$

$$\int_0^{2\pi} \cos(mt)\sin(nt)\,dt = 0$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(mt)\,dt$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(mt)\,dt$$

Fourier Transform

$$F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

inv = $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk$

Dirac Delta

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

&

$$\int_{-\infty}^{\infty} f(x)\,\delta(x) = f(0)$$

$-x_0$ shifts by x_0

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau = f(t)$$

Change of Basis

matrix is $[\vec{b}_1 \text{ in } e, \vec{b}_2 \text{ in } e \ldots] \leftarrow C$

to go from e to b

inv. goes from b to e

Transformation T in e

is $C^{-1}TC$ in b

conv. to basis b

apply transf

conv. back to e

to change a vector, use inv. cob m