PHYSICS 5C Crib Sheet - DIVIT RAWAL

$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2$; $\langle u\rangle\langle v\rangle$ for $u \perp\!\!\!\perp v$.

$Pr[r] = \frac{1}{\Omega(E)}$ if $E_r = E$.

Heat is thermal energy in transit:

$C_V = \left(\frac{\partial Q}{\partial T}\right)_V$; $C_p = \left(\frac{\partial Q}{\partial T}\right)_p$

Thermal Contact:

$E = E_a + E_b \to \Omega(E) = \Omega_a(E_a)\,\Omega_b(E_b)$.

Most probable maximizes $\Omega(E)$:

$\frac{\partial}{\partial E_a}\ln(\Omega_a(E_a)) = \frac{\partial}{\partial E_b}\ln(\Omega_b(E_b))$

$\Rightarrow \frac{1}{k_B T} = \frac{\partial}{\partial E}\ln(\Omega(E))$.

Sys in thermal contact w/ reservoir:

$E_{tot} = E_{sys} + E_{res}$ const.

$Pr[E_s] \propto \Omega_R(E_{tot} - E_s)$. Exp:

$\ln(\Omega_R(E_{tot} - E_s)) \approx \ln(\Omega_R(E_{tot})) - \frac{E_s}{k_B T}$

$\Rightarrow \Omega_R(E_{tot} - E_s) \propto \exp(-\beta E_s)$.

$Pr[r] = \frac{\exp(-\beta E_r)}{Z}$ for $Z = \sum_r \exp(-\beta E_r)$.

Boltzmann Distr.

$E = \frac{1}{2}mv^2 = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2)$.

$Pr[v_x, v_y, v_z] = A\exp(-av_x^2)\exp(-av_y^2)\exp(-av_z^2)$ for $a = \frac{m}{2k_BT}$ for A normalize const

$g(v_x) = C\exp(-av_x^2)$, $C = \sqrt{\frac{a}{\pi}} = \sqrt{\frac{m}{2\pi k_B T}}$

Moments of one component:

$\langle v_x\rangle = 0$ (odd integrand) $\langle v_x^2\rangle = \frac{k_BT}{m}$

$\langle |v_x|\rangle = 2\int_0^\infty v_x g(v_x)\,dv_x = \sqrt{\frac{2k_BT}{\pi m}}$

By indep. $Pr[v_x, v_y, v_z] = \left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp\left(-\frac{m(v_x^2+v_y^2+v_z^2)}{2k_BT}\right)$

Speed Distr:

$f(v)dv = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2k_BT}\right)^{3/2} v^2 dv\, \exp(-mv^2/2k_BT)$.

$\langle v\rangle = \sqrt{\frac{8k_BT}{\pi m}}$ $\langle v^2\rangle = \frac{3k_BT}{m}$

Most Probable Speed $v_{mp} = \sqrt{\frac{2k_BT}{m}}$

$\sqrt{2} < \langle v\rangle < v_{rms} \iff \sqrt{2} < \sqrt{\frac{8}{\pi}} < \sqrt{3}$

Energy: $\Delta U = \Delta Q + \Delta W \equiv dU = dQ + dW$

$dW = -p\,dV$. $dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$

$C_V = \left(\frac{\delta Q}{dT}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V$ $C_p = \left(\frac{\delta Q}{dT}\right)_p = \left(\frac{\partial U}{\partial T}\right)_V + \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p$

$C_p - C_V = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p$

Ideal Gas: $PV = RT$ per mole

$\left(\frac{\partial V}{\partial T}\right)_p = \frac{R}{P}$.

Mayer Reln. $C_p - C_V = R$

Monoatomic Ideal Gas (Eq. partition):

$U = \frac{3}{2}RT$, $C_V = \frac{3}{2}R$, $C_p = \frac{5}{2}R$,

$\gamma = \frac{C_p}{C_V} = \frac{5}{3}$

Isothermal: $\Delta T = 0$ (reversible for ideal). For ideal gas $\to \Delta U = 0 \Rightarrow dQ = p\,dV$

$Q_{iso,rev} = \int_{V_1}^{V_2} p\,dV = RT\ln\left(\frac{V_2}{V_1}\right)$

Heat $= RT\ln\left(\frac{V_2}{V_1}\right)$. $W = -Q_{iso,rev}$ (work done by gas).

Adiabatic: $\delta Q = 0 \Rightarrow dU = dW = -p\,dV$

$U = C_V T$, $p = \frac{RT}{V} \Rightarrow \frac{dT}{T} = -\frac{R}{C_V}\frac{dV}{V}$

Integrate: $\ln\left(\frac{T_2}{T_1}\right) = -\frac{R}{C_V}\ln\left(\frac{V_2}{V_1}\right)$.

Mayer's $\Rightarrow TV^{\gamma-1} = $ const.

$\Rightarrow pV^\gamma = $ const, $Tp^{\frac{1-\gamma}{\gamma}} = $ const.

Carnot Engines.

1. Isothermal Exp: $T_h: A \to B$ (absorbs Q_h, does work).
2. Adiabatic Exp: $B \to C$ $T_h \to T_\ell$, $Q = 0$
3. Isothermal Comp: $T_\ell: C \to D$ (sys rejects heat Q_ℓ)
4. Adiabatic Comp: $D \to A$ $T_\ell \to T_h$ $Q = 0$

$\frac{Q_h}{Q_\ell} = \frac{T_h}{T_\ell}$ $\eta_{carnot} = \frac{W}{Q_h} = 1 - \left(\frac{Q_h}{Q_\ell}\right)^{-1} = 1 - \left(\frac{T_h}{T_\ell}\right)^{-1}$

$\oint \frac{dQ}{T} \geq 0$ (equality iff cycle is reversible). $\int_A^B \frac{dQ}{dT}$ is path indep.

$dS = \frac{\delta Q_{rev}}{T} \Rightarrow (S(A) - S(B)) = \int_A^B \frac{\delta Q_{rev}}{T}$

$\Rightarrow dS = 0$ for adiabatic rev (isentropic)

$dS \geq \frac{\delta Q}{T}$ (= for reversible proc.).

$\delta Q = 0 \Rightarrow dS \geq 0$ $dU = T\,dS - p\,dV$

$dU = \left(\frac{\partial U}{\partial S}\right)_V dS - \left(\frac{\partial U}{\partial V}\right)_S dV$ $T = \left(\frac{\partial U}{\partial S}\right)_V$ $p = -\left(\frac{\partial U}{\partial V}\right)_S$

$S = k_B \ln\Omega$

$S = -k_B\sum_i P_{r_i}\ln P_{r_i}$

$H = U + PV$ $G = U - TS + PV$.

$F = U - TS$

$\left(\frac{\partial U}{\partial V}\right)_T = 0$ Transl: 3 DoF, Rotat: 2 DoF, Vibr: 2 DoF (only @ high T)

$S = k_B\left(\ln Z - \beta\frac{\partial \ln Z}{\partial\beta}\right) = k_B(\ln Z + \beta\langle E\rangle)$. $\langle E\rangle = -\frac{\partial\ln Z}{\partial\beta} = U$

Maxwell Relns. $\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$, $\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$,

$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$, $\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$

$\left(\frac{\partial E}{\partial S}\right)_V = T$

$Var[E] = \frac{\partial^2}{\partial\beta^2}\ln Z$ $F = -k_BT\ln Z$ $p = -\left(\frac{\partial F}{\partial V}\right)_T$ if energies dep. on vol. $= -k_BT\left(\frac{\partial\ln Z}{\partial V}\right)_T$

Core Formulas: $Q = mc\Delta T$ $\Delta U_{iso} = 0$ for ideal gas

$\Delta S_{sys} = \frac{Q_{rev}}{T}$

Isobaric (const p) $W = P\Delta V$

Isochoric (const V) $W = 0$, $Q = \Delta U$

Isothermal (const T) $\Delta U = 0$, $W = nRT\ln\frac{V_f}{V_i}$ (isothermal $PV = $ const.)

Adiabatic $PV^\gamma = $ const, $Q = 0$, $\Delta U = -W$

$COP_H = \frac{Q_H}{W} = \frac{T_H}{T_H - T_C}$ $COP_R = \frac{Q_C}{W} = \frac{T_C}{T_H - T_C}$ $COP_H = COP_R + 1$ $= \eta \times Q_{in}$

$Q_{in} - Q_{out} = W$; in equil. $\dot{Q}_{in} = \dot{Q}_{out}$

Const Heat Cap: $\Delta S = mc\ln\left(\frac{T_f}{T_i}\right)$ $\Delta S = 0 \iff$ rev.

$\Delta U = Q - W$ (Q = heat added to sys, W = work done by sys).

°C + 273.15 = K. For fixed mass of gas $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$

Mixing Problems: $Q_{lost} + Q_{gained} = 0$

$m_a c_a(T_i^a - T_f) = m_b c_b(T_i^b - T_f)$.

$\Delta S_{a_i} = m_a c_a\ln\left(\frac{T_f}{T_i}\right)$ $\Delta S_{tot} = \sum_i \Delta S_i$

$COP_H \times \dot{W} = \dot{Q}_{in}$, $COP_R \times \dot{W} = \dot{Q}_{out}$ @ steady state $\dot{Q}_{in} = \dot{Q}_{out}$

$W = \int P\,dV$ $\eta = 1 - \frac{Q_{out}}{Q_{in}} = 1 - \frac{T_c}{T_h}$

Adiabatic (reversible): $T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1}$

$T\,dS = dE + p\,dV$

Adiabatic Work: $W = \frac{P_2V_2 - P_1V_1}{1-\gamma} = \frac{nR(T_2 - T_1)}{1-\gamma}$

Adiabatic: $\Delta Q = 0$

Reversible: $\Delta S = 0$.

General $\boxed{\Delta U = Q - W}$

$C_p > C_V$

Ideal Gas Exp. into Vacuum

$Q = 0$, $W = 0$, $\Delta U = 0$, T same, $\Delta S > 0$

$P = \frac{F}{A}$

Equipartition of Energy: For f DoF $U = \frac{f}{2}Nk_BT$, $C_V = \frac{f}{2}Nk_B$

$E = \sum_{i=1}^{n} a_i x_i^2$ $a_i > 0$ $\langle E\rangle = \frac{n}{2}k_BT$

$C_p = C_V + Nk_B$, $\gamma = \frac{C_p}{C_V} = 1 + \frac{2}{f}$ $Nk_B \to R$ for per mole

Diatomic: H, N, O, F, Cl, Br, I

Monatomic: He, Ne, Ar, Kr, Xe

Time Dependent Schrödinger Eqn.

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi(x,t)$$

Linear PDE $\Rightarrow$ superposition applies

Born Interpretation

$\Psi(x,t) \in \mathbb{C}$

$Pr[x,t] = |\Psi(x,t)|^2$

$Pr[x \in (a,b)] = \int_a^b |\Psi(x,t)|^2 dx$

Normalization: Only L^2 finite allowed (not $\Psi = 0$).

$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$

If normalized @ $t=0$, stays normalized

Expectation Values

For operator $\hat{Q}$:

$\langle Q \rangle = \int \Psi^*(x,t)\,\hat{Q}\,\Psi(x,t)\,dx$

Position: $\hat{x} = x$ $\quad \langle x \rangle = \int x|\Psi|^2 dx$

Momentum: $\hat{p} = -i\hbar \frac{d}{dx}$ $\quad \langle p \rangle = \int \Psi^* (-i\hbar \frac{\partial}{\partial x}) \Psi dx$

Kinetic Energy: $\hat{T} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

Observables

Every observable $\Longleftrightarrow$ Hermitian operator

$\hat{Q}\phi = q\phi$ Hermitian $\Rightarrow q \in \mathbb{R}$

Time Independent Schrödinger Eqn.

Assume $V(x) = V(x,t)$

Separation of Variables gives

$\Psi(x,t) = \psi(x)\,T(t)$.

$T(t) = e^{-iEt/\hbar}$

$$-\frac{\hbar}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Hamiltonian: $\hat{H} = -\frac{\hbar}{2m}\frac{d^2}{dx^2} + V(x)$

Stationary States:

Form $\Psi_n(x,t) = \psi_n(x)e^{-iE_nt/\hbar}$

$\Rightarrow |\Psi_n|^2 = |\psi_n|^2$. All exp. vals are const in time. Energy is exact.

$R + T = 1$; ψ cont.

ψ' cont. except for δ pot.

Superposition

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_nt/\hbar}$$

where $c_n = \int \psi_n^*(x)\,\Psi(x,0)\,dx$

$Pr[E_n] = |c_n|^2$ s.t. $\sum_n |c_n|^2 = 1$

$\langle H \rangle = \sum_n |c_n|^2 E_n$

Infinite Square Well (0 to a)

$$V(x) = \begin{cases} 0 & x \in (0,a) \\ \infty & \text{else} \end{cases}$$

$\psi(0) = \psi(a) = 0$.

$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ $\quad n \in \mathbb{Z}_{>0}$

$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ $\quad$ n-1 nodes inside well

Harmonic Oscillator

$V(x) = \frac{1}{2}m\omega^2x^2$

$E_n = \hbar\omega(n + \frac{1}{2})$, $n = \mathbb{Z}_{\geq 0}$

$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + i\hat{p})$

$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - i\hat{p})$

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Uncertainty Principle

$\sigma_A\sigma_B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

$\Rightarrow \sigma_x\sigma_p \geq \frac{\hbar}{2}$

Delta Function Barrier

$V(x) = a\,\delta(x-a)$ $\quad k \doteq \sqrt{\frac{2mE}{\hbar^2}}$

ψ cont $\Rightarrow \psi'(a^+) - \psi'(a^-) = \frac{2ma}{\hbar^2}\psi(a)$

$T_{pr} = \frac{1}{1 + \left(\frac{ma}{\hbar^2 k}\right)^2}$ $\quad R_{pr} = \frac{\left(\frac{ma}{\hbar^2 k}\right)^2}{1 + \left(\frac{ma}{\hbar^2 k}\right)^2}$

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & x < a \\ Fe^{ikx} & x > a \end{cases}$$

(ass. left incidence)

$\psi(a^+) = \psi(a^-)$; $R_{pr} = \frac{|B|^2}{|A|^2}$; $T_{pr} = \frac{|F|^2}{|A|^2}$

Finite Sq Well

$$V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & |x| \geq a \end{cases}$$

$k \doteq \sqrt{\frac{2mE}{\hbar^2}}$ $\quad k_2 \doteq \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ Ce^{ik_2x} + Be^{-ik_2x} & |x| < a \\ Fe^{ikx} & x > a \end{cases}$$

ψ cont. @ $\pm a = x$, ψ' cont. @ $x = \pm a$

$R_{pr} = \frac{|B|^2}{|A|^2}$; $T_{pr} = \frac{|F|^2}{|A|^2}$

$E^2 = (pc)^2 + (mc^2)^2$

Math

$\ln n! \approx n\ln n - n$

$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$

$\left(\frac{\partial x}{\partial y}\right)_z = \left[\left(\frac{\partial y}{\partial x}\right)_z\right]^{-1}$ $\quad \left(\frac{\partial x}{\partial z}\right)_w = \left(\frac{\partial x}{\partial y}\right)_w\left(\frac{\partial y}{\partial z}\right)_w$

$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$ $\quad (1+x)^n \approx 1 + nx$ for $|x| \ll 1$

$S_n = a\frac{1-r^n}{1-r}$ for a first in geo sum; $S_\infty = \frac{a}{1-r}$

$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = \int_0^L \cos^2\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}$ $\quad \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$ $(n \neq m)$

$\int_0^L x\sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{L^2}{4}$ $\quad \int_0^L x^2\sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{L^3}{6} - \frac{L^3}{4n^2\pi^2}$

$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, $a > 0$ $\quad \int_0^\infty e^{-2Kx}dx = \frac{1}{2K}$

Wave/Particle Stuff

$\lambda = \frac{h}{p}$ $\quad p = \sqrt{2mK}$ $\quad \hbar = \frac{h}{2\pi}$

Bragg Condition

$2d\sin\theta = n\lambda$ $\quad d_{min} = \frac{\lambda}{2}$

d: spacing b/w crystal planes; n: order of diffraction peak; θ: $\angle$ incident beam & crystal plane

Photoelectric Effect

$E_\gamma = hf = \frac{hc}{\lambda}$ $\quad hf = \phi + K_{max}$ (ϕ: work fn; K_{max}: max K of e^-)

At threshold, $\phi = hf_0 = \frac{hc}{\lambda_0}$

$K_{max} = hf - \phi = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$. $\quad eV_s = K_{max}$ $\quad \lambda > \lambda_0 \Rightarrow$ no e^-

$p = \hbar k$, $E = \frac{\hbar^2k^2}{2m}$

Photons: $E = hf = \hbar\omega$ $\quad p = \frac{E}{c} = \frac{h}{\lambda}$

$v_{phase} = \frac{\omega}{k}$ (individual wave crests) $\quad v_{grp} = \frac{d\omega}{dk}$ (physical velocity) $\quad v_{phase} = \frac{v_{grp}}{2}$ for free particle

Rectangular Barrier

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 < x < a \\ 0 & x > a \end{cases}$$ $\quad k \doteq \frac{\sqrt{2mE}}{\hbar}$

$\psi_I = Ae^{ikx} + Be^{-ikx}$ $\quad \psi_{III} = Fe^{ikx}$

$j = \frac{\hbar k}{m}|A|^2$

$T \propto e^{-2Ka}$

If $E > V_0$: $q \doteq \frac{\sqrt{2m(E-V_0)}}{\hbar}$

$\psi_{II} = Ce^{iqx} + De^{-iqx}$

If $E < V_0$: $K \doteq \frac{\sqrt{2m(V_0-E)}}{\hbar}$

$\psi_{II} = Ce^{Kx} + De^{-Kx}$

Qualitative Stuff: oscillatory $\Leftrightarrow$ classically allowed, exponential $\Leftrightarrow$ classically forbidden

Node $\doteq \psi = 0$. nth bound state = $n-1$ nodes, more nodes $\Rightarrow$ more E

Bound: $E < V$ Scattering: $E > V$

For perfect transmission $E > V$, $qL = n\pi$ $\quad q \doteq \frac{\sqrt{2m(E-V)}}{\hbar}$

Transmission

$E > V_0$ $\quad T = \left(1 + \frac{V_0^2}{4E(E-V_0)}\sin^2(qL)\right)^{-1}$

$E < V_0$ $\quad T = \left(1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2(KL)\right)^{-1}$