$\vec{a}\cdot\vec{b} = \cos\theta|a||b| = \vec{b}\cdot\vec{a}$, $\vec{a}\cdot\vec{a} = |a|^2$

$\hookrightarrow = 0$ if $\vec{a}\perp\vec{b}$

$= a_x b_x + a_y b_y + a_z b_z$

$|\vec{a}\times\vec{b}| = |a||b|\sin\theta$, direction using RHR

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

$\hat{i}\times\hat{j} = \hat{k}$
$\hat{j}\times\hat{k} = \hat{i}$
$\hat{k}\times\hat{i} = \hat{j}$

Divit Rawal

DIMENSIONAL ANALYSIS

CONSERVATION OF STRING

$v = \frac{dr}{dt}$

$a = \frac{d^2r}{dt^2}$

Max & Min usually $\Rightarrow$ different directions of friction

F on mass due to shell (spherical) = same as point mass of same mass

$v = \dot{x}$

$a = \dot{v} = \ddot{x}$

$\frac{d(A\cdot B)}{dt} = \frac{dA}{dt}\cdot B + \frac{dB}{dt}\cdot A$

$\omega = \frac{d\theta}{dt} = \frac{v}{r}$

$\frac{d}{dt}(A\times B) = \frac{dA}{dt}\times B + A\times\frac{dB}{dt}$

$v(t) = v_0 + \int_{t_0}^{t} a(t')\,dt$

$F_{Hooke} = -kx$

$\omega = \sqrt{\frac{k}{m}}$

$v(t) = v_0 + at$

$r(t) = r_0 + \int_0^t (v_0 + at')\,dt'$

$= r_0 + v_0 t + \frac{at^2}{2}$

assuming constant acceleration

$v_i^2 = v_{0i}^2 + 2a_i\Delta x_i$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

$\hat{r}, \hat{\theta}$ vary w/ position, directions only depend on θ $\Rightarrow \hat{r}(\theta), \hat{\theta}(\theta)$

$x = r\cos\theta$
$y = r\sin\theta$

vectors can change w/ magnitude or direction

$r = r\hat{r}(\theta)$

$\Rightarrow \frac{dr}{dt} = \frac{dr}{dt}\hat{r}(\theta) + r\frac{d\hat{r}(\theta)}{dt}$

$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$ $\quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$

$\Rightarrow v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\Rightarrow a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} + \ddot{z}\hat{z}$

$F_b = -F_a$

$|F_{b,a}| = \frac{GM_aM_b}{r^2}$

$G = 6.673\times10^{-11}$

$|F_{b,a}| = -\frac{GM_aM_b}{r^2}\hat{r}_{b,a}$

$F_{b,a} = -F_{a,b}$

if R is the vector from the origin of an inertial system to another

$F_{apparent} = F_{true} \underbrace{- M_s\ddot{R}}_{\text{fictitious forces}}$

Viscous forces

$F_v = -Cv$

$m\frac{dv}{dt} = -Cv$

$F_s = -k(x - x_0)$

$\left.\frac{d^2x}{dt^2} + \omega^2 x = 0\right]$ SHM

$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0$

$x = C\sin(\omega t + \varphi)$

$= C\sin\omega t\cos\varphi + C\cos\omega t\sin\varphi$

Taylor Series

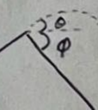
$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots \approx 1 + nx$ for $x \ll 1$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = 1 + x \left(+\frac{x^2}{2!}\right)$ for $x \ll 1$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = x - \frac{x^3}{3!}$ for $x \ll 1$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = 1 - \frac{x^2}{2!}$ for $x \ll 1$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots = x - \frac{x^2}{2}$ for $x \ll 1$



Range $= \frac{2v_0^2\cos^2\theta(\tan\theta + \tan\varphi)}{g}$

$x^* \tan\varphi$

$F = \frac{\partial p}{\partial t} = ma$

$\frac{dP}{dt} = M\ddot{R}$ (center of mass)

For rocket-type problems, consider momentum just before & after an event (ie dv & dm)

Uniform Circular Motion $\Rightarrow a = -r\dot{\theta}^2\hat{r}$

$= -\omega R$

$= \left|\frac{v^2}{R}\right|$

$\frac{2\pi}{T} = \omega$

b/c $\omega r = v$

$\Delta s = R\Delta\theta = vT$

$R = \frac{1}{M}\sum_i m_i r_i$

$= \frac{1}{M}\int dm\, r$

$= \frac{1}{M}\int dV \frac{dm}{dV} r$

$= \frac{1}{M}\int dV\, r\rho(r)$

$F = \frac{dp}{dt} = \dot{P}_{in} - \dot{P}_{out}$

$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2+y^2}}$

$\hat{\theta} = \frac{-y\hat{x} + x\hat{y}}{\sqrt{x^2+y^2}}$

$M\ddot{R}_{CM} = F_{tot\,ext}$

$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$

Rolling w/o slipping

$\Delta\theta = \frac{x}{R}$

$\Delta s = \Delta x$

$x = r\theta$

Elastic Collisions

$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$

Inelastic

$m_1v_1 + m_2v_2 = Mv$

$\sin\theta = \cos\left(\theta + \frac{\pi}{2}\right)$

Newton's Third Law

DRAW ALL FBD's

and Second and First

Momentum

$\vec{p} = m\vec{v}$

$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt}$

$\frac{dM}{dt} = -\frac{dm}{dt}$ For Rocket-type problems, consider $P(t)$, $P(t+dt)$

$\frac{d}{dt}\log(x) = \frac{\dot{x}}{x}$, consider

$$Com = \frac{1}{M}\sum_i m_i r_i = R = \frac{1}{M}\int r\,dm \begin{cases} \frac{1}{M}\int x\lambda\,dx & \text{1-D},\ \lambda = \frac{dm}{dx} \\ \frac{1}{M}\int r\sigma\,dA & \text{2-D}\ \ \sigma = \frac{dm}{dA} \\ \frac{1}{M}\int r\rho\,dV & \text{3-D}\ \ \rho = \frac{dm}{dV} \end{cases}$$

$M\ddot{R} = F_{ext}$

Flux assume density $dm/dx = \rho$

$\frac{\text{momentum}}{\text{length}} = \frac{dm\vec{v}}{dx} = \rho A_\perp v$ ($A_\perp$ cross-sectional area) $\Rightarrow \frac{\text{momentum}}{\text{time}} = \frac{\text{momentum}}{\text{length}} \cdot \frac{\text{length}}{\text{time}} = \rho A_\perp v^2$

$P_f = P_0 + F\Delta t$

Momentum is always conserved in collisions

Energy

$W_F = \int_\gamma F \cdot dx$ ← depends on path

only nonconservative force is friction

$KE = \frac{mv^2}{2} = \frac{MV^2}{2}$

$F = -\nabla U \Rightarrow U = -\int_\gamma F \cdot dx$

$W_F = \int_a^b F \cdot dx = U_a - U_b$

↳ conservative force, path-independent

$W_{tot} = \int_\gamma F_{tot} \cdot dx = \int_\gamma ma \cdot dx = m\int_\gamma a \cdot v\,dt = m\int \frac{d}{dt}(KE)\,dt = \Delta KE$

$W_{tot} = KE_b - KE_a = U_a - U_b \Rightarrow KE_a + U_a = KE_b + U_b$

$\Rightarrow$ cons. of mech. energy

Force	F	U	Ref Pt.
constant	$-C\hat{x}$	Cx	$r = 0$
Gravity	$-\frac{GMm}{r^2}\hat{r}$	$-\frac{GMm}{r}$	$r = \infty$
Gravity (Earth)	$-mg\hat{z}$	$-mgh$	$r = 0$
Spring	$-kx$	$\frac{1}{2}kx^2$	$r = 0$

Stability

$\frac{dU}{dx} = 0,\ \frac{d^2U}{dx^2} > 0 \Rightarrow$ stable eq

$\frac{dU}{dx} = 0,\ \frac{d^2U}{dx^2} < 0 \Rightarrow$ unstable eq.

$\frac{dU}{dx} = 0,\ \frac{d^2U}{dx^2} = 0 \Rightarrow$ normal eq.

2-D Rotation

$v_\phi = \omega \times r$

$L = r \times p = L\hat{z}$

$\tau = r \times F$

$\frac{dL}{dt} = \frac{dr}{dt} \times p + \frac{dp}{dt} \times r = v \times mv + r \times \frac{dp}{dt} = r \times F = \tau$

$L = \sum_i r_i \times p_i = \sum_i (r_i\hat{r}) \times (m_i \omega r_i \hat{\phi}) = \sum_i m_i r_i^2 \omega \hat{z} = I\omega\hat{z}$

$I = \sum_i m_i r_i^2$
$= \int r^2\,dm = \int (x^2+y^2)\,dm$
$= \begin{cases} \int x^2 \lambda\,dm & \text{1-D} \\ \int (x^2+y^2)\sigma\,dm & \text{2-D} \\ \int (x^2+y^2)\rho\,dm & \text{3-D} \end{cases}$

for axes through cm

Shape	I
pt. mass	mr^2
pt. masses	$\sum m_i r_i^2$
Rod	$\frac{1}{12}ml^2$
Ring	mr^2
Disk	$\frac{mr^2}{2}$
sphere shell	$\frac{2mr^2}{3}$
sphere	$\frac{2}{5}mr^2$

$KE_{rot} = \frac{I\omega^2}{2}$

$L = \underbrace{R \times P}_{\text{rotation of COM}} + \underbrace{I_{cm}\omega}_{\text{rot ab. COM}}$

$\Rightarrow L = (I_{cm} + MR^2)\omega$ ← parallel axis thm.

$\tau_{cm} = I_{cm}\alpha$

any momentum conserved iff pivot is ref. pt

3-D Rotation

Tensor of Inertia

$$\int \begin{bmatrix} y^2+z^2 & -xy & -zx \\ -xy & z^2+x^2 & -yz \\ -zx & -yz & x^2+y^2 \end{bmatrix} dm$$

$\omega = \omega_s + \Omega\hat{z}$

$\frac{dL}{dt} = \Omega \times L$

$\Omega = \frac{mlg}{L_s} = \frac{mlg}{I_s\omega_s} = \frac{lg\cos\theta}{L_s\cos\theta}$

Periodic Motion

$F = -\frac{dU}{dx} = -kx$

$U(x) \approx U_0$

$x(t) = A\cos(\omega t + \phi) = Re(e^{i\omega t + \phi})A$ — phase

amplitude, ang. freq $\rightarrow \sqrt{\frac{k}{m}}$

$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$ $\quad \phi = -\arctan\left(\frac{v_0}{\omega x_0}\right)$

for $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$

where $\gamma = b/m$ & $\omega = \sqrt{k/m}$

if $\omega_0 \gg \gamma/2$, sol is

$x(t) = X_0 e^{-\alpha t}\cos(\omega_1 t + \phi)$

$\alpha = \gamma/2$ $\quad \omega_1 = \sqrt{\omega_0^2 - (\gamma/2)^2}$ & X_0, ϕ determined by init. conditions

for $\omega_0 \gg \gamma/2$, $v = -\omega_0 X_0 e^{-\gamma/2 t}\sin(\omega_1 t + \phi)$

Damped-Driven Motion

$E = \frac{kX_0^2 e^{-\gamma t}}{2}$

$Q = \frac{\text{avg energy in oscillator}}{\text{avg energy dissipated in 1 rad}}$

$\Delta E = \gamma E$

$\Rightarrow Q = \frac{\omega_0}{\gamma}$ — ω_0, decay rate inv.

$F_{drive}(t) = F_0\cos(\omega t)$

$x = X_0\cos(\omega t + \phi) \Rightarrow \ddot{x} + \gamma\dot{x} + \omega^2 x = \frac{F_0}{m}\cos\omega t$

$X_0 = \frac{F_0}{m}\frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\omega\gamma)^2}}$, $\phi = -\arctan\left(\frac{\omega\gamma}{\omega^2 - \omega_0^2}\right)$

amp. max when $\omega = \omega_0$ or $\omega \approx \omega_0$ resonance

$\Rightarrow X_0 = \frac{F_0}{2m\omega_0}\left(\frac{1}{\sqrt{(\omega - \omega_0)^2 + (\gamma/2)^2}}\right)$

$\phi = \arctan\left(\frac{\gamma/2}{\omega - \omega_0}\right)$

$\Rightarrow E = \frac{k^2 X_0^2}{2}$

@ peak $E(\omega) = \frac{1}{(\omega - \omega_0)^2 - \frac{\gamma}{2}}$ ← lineshape

resonance width (FWHM) @ $g(\omega)$ half of max $\omega = \omega_0 \pm \gamma/2$

$\frac{\text{resonance freq}}{\text{freq FWHM}} = \frac{\omega_0}{\gamma} = Q$ — size of band of resonance freqs. inv

general sol to damped-driven includes sol to damped w/o driving

Superposition

same frequency

$x_1 = A_1\cos(\omega t + \alpha_1)$

$x_2 = A_2\cos(\omega t + \alpha_2)$

$x = x_1 + x_2 = A\cos(\omega t + \phi)$

$\Rightarrow A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_1 - \alpha_2)$, $\alpha = \alpha_1 + \beta$

where $A\sin\beta = A_2\sin(\alpha_2 - \alpha_1)$

let $\delta = \alpha_2 - \alpha_1$

$\Rightarrow x = e^{i(\omega t + \alpha_1)}(A_1 + A_2 e^{i\delta})$

for N waves w/ freq. ω & relative phase δ

$X = A_0\frac{\sin(N\delta/2)}{\sin(\delta/2)}\cos\left(\omega t + \frac{N-1\delta}{2}\right)$

Beats

iff 2 waves are commensurable, $n_1T_1 = n_2T_2$ $\quad n_1, n_2 \in \mathbb{Z}$, form a periodic fn w/ $T = n_1T_1$

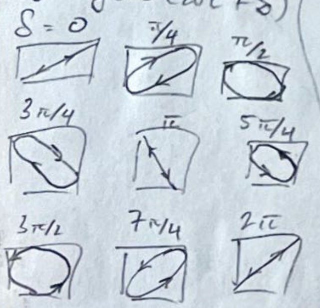
if two waves have very similar freq., they combine w/ $T = (T_1 + T_2)/2$, & varying amplitude

if $A_1 = A_2 \Rightarrow 2A\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)$

Perpendicular Motion

$x = A_x\cos(\omega t)$

$y = A_y\cos(\omega t + \delta)$



Complex Number Stuff

$e^{i\theta} = \cos\theta + i\sin\theta$

$z = |z|e^{i\phi}$

$z_1z_2 = |z_1||z_2|e^{i(\phi_1 + \phi_2)}$

Lorentz Transformation

$t' = \gamma(t - \frac{vx}{c^2})$

$x' = \gamma(x - vt)$

$y' = y$

$z' = z$

A on ground, B @ v, throws ball w/ u'

$u_x = \frac{v + u'_x}{1 + \frac{vu'_x}{c^2}}$ where $\gamma > 1$

$u_y = \frac{u'_y}{\gamma(1 + \frac{vu'_x}{c^2})}$

$u_z = \frac{u'_z}{\gamma(1 + \frac{vu'_x}{c^2})}$

Relativistic Doppler

@ freq ν_s

$\nu_{observer} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}\cos\theta}\nu_s$

if $\theta = 0$,

$\nu_0 = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\nu_s$ $\quad \tan\theta' = \frac{u'_y}{u'_x}$

τ - proper time

$P = (E/c, \vec{P})$

$\Rightarrow E^2 - p^2c^2 = (m_0c^2)^2$ ← invariant in all frames

for $m_0 = 0$

$E = pc$ $\quad p_{photon} = \frac{h\nu}{c}$ — Plank's constant, ν — freq

4-Momentum is conserved for relativistic collisions

Relativity

$\Delta t' = \gamma\Delta t$, $\gamma = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$

$L' = \frac{L}{\gamma}$

Invariant Relativistic ~~Dot~~ Inner Prod

$U \cdot W = u^0w^0 - u^1w^1 - u^2w^2 - u^3w^3$

same in all frames

Invariant Spacetime Interval

position 4-vector: $X = (ct, x, y, z)$

$\Delta s^2 = c^2\Delta t^2 - \Delta\vec{x}^2$

timelike - $\Delta s^2 > 0$, some frame w/ two events @ diff time, same loc

spacelike - $\Delta s^2 < 0$, some frame w/ same time, diff loc

lightlike - $\Delta s^2 = 0$

4-velocity

$U = \gamma(c, \vec{v})$ $\quad \vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ lab's frame

4-momentum

$P = m_0U = \gamma(cm_0, c\vec{v})$

$P \cdot P = (m_0c)^2$

4-acceleration

$A = \frac{dU}{d\tau} = \gamma\frac{dU}{dt} = (\gamma\dot{\gamma}c, \gamma\dot{\gamma}\vec{v} + \gamma^2\vec{a})$

4-Force

$F = m_0A$

Normal Mode - whole system oscillates @ same freq

$m\ddot{x}_1 = -k(x_1 - x_2)$

$m\ddot{x}_2 = -k(x_2 - x_1)$ $\quad$ ansatz $x = A_i e^{i\omega t}$

$\Rightarrow \begin{bmatrix} -m\omega^2 + k & -k \\ -k & -m\omega^2 + k \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \det = 0$ iff $A_1, A_2 \neq 0$

lg $Q \Rightarrow$ small energy decay

lg $Q \Rightarrow$ selective band of freqs that excite resonance

$x = x^h + x^p$

h: $\ddot{x}_h + \gamma\dot{x}_h + \omega^2x_h = 0$

decays over time

n-h: $\ddot{x}_p + \gamma\dot{x}_p + \omega^2x_p = F_0\cos(\omega t)$

$\ddot{x}_h + \gamma\dot{x}_h + \omega^2x_h + \ddot{x}_p + \gamma\dot{x}_p + \omega^2x_p = F_0\cos(\omega t)$

Damping

Consider $a\ddot{y} + b\dot{y} + cy = 0$

$y \sim Ae^{rt}$

$\Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Overdamped ($b^2 > 4ac$)

$y(t) = Ae^{\frac{-b + \sqrt{b^2 - 4ac}}{2a}t} + Be^{\frac{-b - \sqrt{b^2 - 4ac}}{2a}t}$

does not oscillate

Underdamped ($4ac > b^2$)

$y(t) = Ce^{-bt/2a}\cos(\omega' t + \phi)$ $\quad \omega' = \frac{\sqrt{|4ac - b^2|}}{2a}$

for C, ϕ in terms of A & B

$A = A_+ - A_-$ $\quad B = i(A_+ - A_-)$

oscillates

Critically Damped ($b^2 = 4ac$)

$y(t) = (A + Bt)e^{-\frac{bt}{2a}}$

$= Ae^{-bt/2a}\cos(\omega' t) + Be^{-bt/2a}\sin(\omega' t)$