Line Integrals

(1) Scalar fn w.r.t. arc len

$\int_C f\,ds = \int_a^b f\,ds$

(2) Scalar fn w.r.t. x, y, z...

$\int_C f\,dx = \int_a^b f\,d\,x'(t)\,dt$

3) Vec Field (work)

$\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt$

FTLI:

$\int_C \nabla f\cdot dr = f(r(b))\,f(r(a))$

$\nabla\times F = 0 \Leftrightarrow F$ cons $\Leftrightarrow F = \nabla f$

curl grad $= 0$

div curl $= 0$

Stoke's Thm. (C +ively oriented) CCW

$\int_C F = \iint_S \text{curl}\,F\,dS$

$F\cdot dr = \iint_S \nabla\times F\cdot n\,dS$

Green's Thm (simple, closed)

$\int_C P\,dx + Q\,dy = \iint_D Q_x + P_y\,dA$

Div. Thm.

$\iiint_V \nabla\cdot F = \iint_S F\cdot n\,dS = \iint_S F\,dS$

Flux $= \iint F\,dS$

$\iint_S F\cdot n\,dS = \iint_D F\cdot(r_u\times r_v)\,dA$

Curl $= \nabla\times F$ Normal vec to

Div $= \nabla\cdot F$ tang plane is

$|r_u\times r_v|$

$A(S) = \iint_D |r_u\times r_v|\,dA = \iint_D \sqrt{1+z_x^2+z_y^2}\,dA$

$\text{Vol(Sphere)} = \frac{4\pi R^3}{3}$

$\text{SA(Sphere)} = 4\pi R^2$

$\text{Vol(Ellipsoid)} = \frac{4}{3}\pi abc$

$\text{Vol(Cone)} = \frac{\pi R^2 h}{3}$

SA (cone) w/ base $= \pi R\sqrt{R^2+h^2} + \pi R^2$

$\int_C f\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$

Tricks:

- order of int.
- diff coords
  - polar, cyl, sphere
  - u,v so over a box

normal to a sphere $\langle x, y, z\rangle$

$\int_C f(x,y)\,ds$

$= \int_a^b f(x(t), y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$

Flux cont $= \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)dA$

$= \iint_D f(r(u,v))\,(r_u\times r_v)\,dA$

$= \iint_D f(x, y, g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$

Order of Int. can simplify calc.

$a\times b = -b\times a$

$(ca)\times b = a\times(cb)$

$a\times(b+c) = a\times b + a\times c$

$a\cdot(b\times c) = (a\times b)\cdot c$

$a\times(b\times c)$

$= (a\cdot c)\,b$

$-(a\cdot b)\,c$

$a\parallel b \Rightarrow a\times b = 0$

$r = \langle x(u,v), y(u,v), z(u,v)\rangle$

$\boxed{n = r_u\times r_v}$

Split bounds into 2 integs when changing

Triangle $u = x,\ v = y,\ z =$ convex comb.

$\int_C f\,ds = \int_a^b f(r(t))\,\|r'(t)\|\,dt$

Center of Mass

$m = \iiint_E \rho\,dV$

$M_{yz} = \iiint_E x\rho\,dV$

$\bar{x} = M_{yz}/m$

$\text{COM} = (\bar{x}, \bar{y}, \bar{z})$

Integrals over surfaces:

param $x(u,v), y(u,v), z(u,v)$

Special: $z = f(x,y)$ $|r_u\times r_v| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}$

Special: surf. part of sphere $|r_\phi\times r_\theta| = R^2\sin\theta$

$u = x+y/2$, $v = x-y/2$ common

$a\times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$L = \int_a^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt$ $A = \int_\alpha^\beta y(t)\, x'(t)\, dt$ SA by rev. $= \int_\alpha^\beta 2\pi y\, dS$ Jacobian $= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$D_u f(x,y) = \nabla f(x,y) \cdot u$ $\geq 0 \rightarrow$ add to integrand when changing coords

$L = \int_a^b \sqrt{r^2 + r'(\theta)^2}\, d\theta$ $A = \frac{1}{2}\int_a^b r^2\, d\theta$ max $|\nabla f|$ when $u \parallel \nabla f$ $\vec{T}(t) = \vec{r}'(t)$

cart $\rightarrow$ polar/cyl $\rightarrow J = r$

max/min check boundary

cart $\rightarrow$ spherical $\rightarrow J = \rho^2 \sin\phi$

$D = f_{xx} f_{yy} - f_{xy} f_{yx}$

Lagrange Multipliers: $g(x,\ldots) = k$ $\nabla f = \lambda \nabla g + \mu \nabla h$

$h(x,\ldots) = l$ solve for λ, sub.

$D < 0 \Rightarrow$ saddle

$D > 0,\ f_{xx} > 0 \Rightarrow$ loc min

$D > 0,\ f_{xx} < 0 \Rightarrow$ loc max

$ax + by + cz + d = 0$

$\rightarrow n = \langle a, b, c\rangle$

Dist. from pt. to plane $= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$

Vol. of Parallelpiped: $|a \cdot (b \times c)|$

Arc len $= \int_a^b \|r'(t)\|\, dt$

Line: $r = r_0 + td$

dist$(p, r) = \frac{\|(p - r_0) \times d\|}{\|d\|}$

$K = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ $y = mx$ diff L (approach indices)

Tang. Plane $z = f(x,y)$ @ (x_0, y_0, z_0)

$(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Tang. Plane $F(x,y,z) = 0 \Rightarrow \nabla F(x_0, y_0, z_0) \perp F(x_0, y_0, z_0)$

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

$\frac{\partial y}{\partial x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ if $\frac{dx}{dt} \neq 0$

Tang. of Polar $= \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

Account for corners!!

Coords:

Spherical

$\rho^2 = x^2 + y^2 + z^2$

$\tan\theta = \frac{y}{x}$

$\cos\phi = \frac{z}{\rho}$

$y = \rho\sin\phi\sin\theta$

$x = \rho\sin\phi\cos\theta$

Polar/Cyl

$r^2 = x^2 + y^2$

$x = r\cos\theta$

$y = r\sin\theta$ $\tan\theta = \frac{y}{x}$

Surfaces (Quadr)

- ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- elliptic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- hyperbolic parab. $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
- cone $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
- hyperboloid (1 sheet) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$
- hyperboloid (2 sheet) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 + \frac{z^2}{c^2}$

Type I: $y(x)$ Type II: $x(y)$

Trig

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

Roses: $r = a\cos(n\theta)$, $a\sin(n\theta)$

odd: petals $= n$, even: petals $= 2n$

Cardiod/Limacon: $r = a \pm b\cos\theta$, $r = a \pm b\sin\theta$

For ellipses, choose x, y so unit circ.

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