PHYSICS 137B Crib Sheet - DIVIT RAWAL

Density Operators

Pure: $\rho = |\psi\rangle\langle\psi| \iff \rho^2 = \rho \iff Tr(\rho) = 1 = Tr(\rho^2)$

Mixed: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \iff \rho^2 \neq \rho \iff Tr(\rho^2) < 1$

s.t. $\sum_i p_i = 1$

$\langle A\rangle = Tr(A\rho) = Tr(\rho A)$; $\rho^\dagger = \rho$; ρ PSD; $Tr(\rho) = 1$

Entropy $S = -Tr(\rho \ln(\rho)) = 0$ iff ρ pure

$i\hbar \frac{d\rho}{dt} = [H, \rho]$. Let $A = \sum_j a_j P_j$. If we obs. a_k, $\rho(t') = \frac{P_k \rho(t) P_k^\dagger}{Tr(P_k \rho(t) P_k^\dagger)} = \frac{P_k \rho(t) P_k^\dagger}{Tr(P_k \rho(t))}$

PVMs $\leftarrow 2 = 0, 1$

$P_j = P_j^\dagger = P_j^2$; $P_j P_k = \delta_{jk} P_j$; $\sum_j P_j = I$

POVMs

B_k PSD, not necessarily Hermitian

$\sum_k B_k^\dagger B_k = I$ $Pr_k = Tr(B_k \rho B_k^\dagger)$

$\rho(t') = \frac{B_k \rho(t) B_k^\dagger}{Tr(B_k^\dagger B_k \rho(t))}$ $\langle\phi|B_k^\dagger B_k|\phi\rangle = \|B_k|\phi\rangle\|^2 \geq 0$

$(B_k^\dagger B_k)^\dagger = B_k^\dagger B_k \doteq E_k$

Just applying ~~read~~ measurement, not reading result, $\rho(t) = \sum_k B_k \rho B_k^\dagger$

$\rho_f = \sum_j \sum_k C_j (U_L (B_k \rho B_k^\dagger) U_L^\dagger) C_j^\dagger$

$P_r(j, k) = Tr(C_j (U_L (B_k \rho B_k^\dagger) U_L^\dagger) C_j^\dagger)$

Quantum "Logic"

$P \wedge Q = Span(Im(P) \cup Im(Q))$

$P \vee Q = Im(P) \cap Im(Q)$

$U = Im(P)$; $U^\perp \doteq \{v \in \mathcal{H} \mid \langle v, u\rangle = 0 \ \forall u \in U\}$

$(U^\perp)^\perp = U$ s.t. $P^\perp = I - P$

De Morgan: $(A \wedge B)^\perp = A^\perp \vee B^\perp$; $(A \vee B)^\perp = A^\perp \wedge B^\perp$

Distr. Law does not hold in general.

Reading CG Table

Find $\langle j_1, m_1, j_2, m_2 | J M\rangle$

$j_1, j_2 \to$ Table; $m_1, m_2 \to$ Row; $J, M \to$ col

$L_\pm |l, m\rangle = \hbar\sqrt{(l \mp m)(l \pm m + 1)}\,|l, m \pm 1\rangle$

$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$ H commutes w/ symmetries

$L_i = \epsilon_{ijk} x_j p_k$

$\frac{\partial\rho}{\partial t} = \frac{1}{i\hbar}[H, \rho]$

Tensor Products

$U \otimes V = span(|u_j\rangle \otimes |v_k\rangle)$

$dim(U \otimes V) = dim U \cdot dim V$

Entangled $\Rightarrow$ not factorizable in any basis

$(\langle a| \otimes \langle b|)(|u\rangle \otimes |v\rangle) = \langle a|u\rangle \langle b|v\rangle$

$(|a\rangle \otimes |b\rangle)^\dagger = \langle a| \otimes \langle b|$

$F \doteq U \otimes V \to U \otimes V$

$Tr_V(F): U \to U$

$Tr_{\cancel{V}}(F) = \sum_{j,k} \langle u_j, v_k | F | u_j, v_k\rangle$

$Tr_V(F) = \sum_k \langle I \otimes v_k | F | I \otimes v_k\rangle$

$Tr_V(A_U \otimes B_V) = A_U\, Tr(B_V)$

$Tr_U(Tr_V(F)) = Tr_V(Tr_U(F)) = Tr(F)$

$Tr(A_U \otimes B_V) = Tr_U(A_U) \otimes Tr_V(B_V)$

Angular Momentum

$\vec{L} = \vec{r} \times \vec{p} = -i\hbar(\vec{r} \times \vec{\nabla})$

$[J^2, J_z] = 0$ $J^2 |j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$

$J_z |j, m\rangle = \hbar m |j, m\rangle$; $j \in \{0, \frac{1}{2}, 1, \ldots\}$

$m \in \{-j, -j+1, \ldots, j-1, j\}$ ($2n+1$ total)

$J_\pm = J_x \pm i J_y$; $[J_+, J_z] = -\hbar J_+$

$[J_-, J_z] = \hbar J_-$ $[J_+, J_-] = 2\hbar J_z$

$(j_1, m_1) \otimes (j_2, m_2) \to (J, M)$

$|j_1 - j_2| \leq J \leq j_1 + j_2$

Interpretations & No-Go Thms.

Locality: spatially sep. $\Rightarrow$ cannot affect each other

Realism: props. exist indep. of measurement.

No-Go: If meas. commute, they have well-def values simoultaneously. If H.V.s exist, all commuting obs. should be predetermined. Contradict by using diff. combs. of observables, expect same, show diff.

Bloch Sphere

$\rho = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma})$; $\langle\vec{\sigma}\rangle = Tr(\rho\vec{\sigma}) = \vec{a}$

Misc.

$\hat{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx$

H atom $E_1 = -\frac{1}{2}\frac{m_e e^4}{\hbar^2}$

$\hookrightarrow \langle V\rangle = 2E_n$

spon. decay to same or lower n

Isospin

E-states $|I, I_3\rangle$, $I \in \{0, \frac{1}{2}, 1, \ldots\}$

$I_3 \in \{-I, -I+1, \ldots, I-1, I\}$ ($2n+1$ tot.)

$|p\rangle \propto |\frac{1}{2}\ {+\frac{1}{2}}\rangle = |uud\rangle$

$|n\rangle \propto |\frac{1}{2}\ {-\frac{1}{2}}\rangle = |udd\rangle$

$|\pi^+\rangle \propto |1\ {+1}\rangle = |u\bar{d}\rangle$

$|\pi^0\rangle \propto |1\ 0\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$

$|\pi^-\rangle \propto |1\ {-1}\rangle = |d\bar{u}\rangle$

$|u\rangle \propto |\frac{1}{2}\ \frac{1}{2}\rangle$ $|\bar{u}\rangle \propto |\frac{1}{2}\ {-\frac{1}{2}}\rangle$

$|d\rangle \propto |\frac{1}{2}\ {-\frac{1}{2}}\rangle$ $|\bar{d}\rangle \propto |\frac{1}{2}\ {+\frac{1}{2}}\rangle$

$[H, I] = 0$ $I_1 \cdot I_2 = \frac{1}{2}(|I_1 + I_2|^2 - |I_1|^2 - |I_2|^2)$

$\Delta H = \frac{1}{4}$ $\Delta H \propto$

$\Delta H = -\frac{3}{4}$ $\hookrightarrow \Delta$ energy for coupled particles

triplet, singlet resp.

$|1\,1\rangle = |pp\rangle$ $|1\,{-1}\rangle = |nn\rangle$

$|1\,0\rangle = \frac{1}{\sqrt{2}}(|pn\rangle + |np\rangle)$

$|0\,0\rangle = \frac{1}{\sqrt{2}}(|pn\rangle - |np\rangle) = |d\rangle$

Spin Stuff

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$,

$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$

$\sigma_i^2 = I$ $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$

$\Rightarrow \sigma_i \sigma_j = -\sigma_j \sigma_i$

$|\pm\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z \pm |-\rangle_z)$

$|\pm\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle_z \pm i|-\rangle_z)$

$S_x = \frac{\hbar}{2}(\sigma_x) \equiv \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$S_y = \frac{\hbar}{2}(\sigma_y) \equiv \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$S_z = \frac{\hbar}{2}(\sigma_z) \equiv \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$S^2 = \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Applying σ_i rotates $180°$ about axis i.

General QM

$(\Delta A)(\Delta B) \geq \left|\frac{1}{2i}\langle[A, B]\rangle\right|$

$\hat{x}|\psi\rangle \to x_i|\psi\rangle$ $\hat{p}|\psi\rangle \to -i\hbar\frac{\partial}{\partial x}|\psi\rangle$

$[A, BC] = [A, B]C + B[A, C]$

$[AB, C] = A[B, C] + [A, C]B$

$\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

$\epsilon_{ijk} = \begin{cases} +1 & \text{if fwd. cycle} \\ -1 & \text{if back. cycle} \\ 0 & \text{else} \end{cases}$

TIPT

Non-Degen:

$$H = H_0 + V;\ H_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle$$

Corrections:

$$E_n^{(1)} = \langle \psi_n^{(0)}|V|\psi_n^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle \psi_m^{(0)}|V|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$|\psi_n^{(1)}\rangle = \sum_{m\neq n} \frac{\langle \psi_m^{(0)}|V|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}|\psi_m^{(0)}\rangle$$

Degen: $\{|\psi_1^{(0)}\rangle, |\psi_2^{(0)}\rangle, \dots, |\psi_g^{(0)}\rangle\}$

$$H^{(1)}_{ij} = \langle \psi_i^{(0)}|V|\psi_j^{(0)}\rangle$$

Solve $H^{(1)}\vec{c}^{(n)} = E_n^{(1)}\vec{c}^{(n)}$

$$|\psi_n^{(0)}\rangle_{corr} = \sum_{i=1}^{g} c_i^{(n)}|\psi_i^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{m\notin \text{degen. subsp.}} \frac{|\langle \psi_m^{(0)}|H^{(1)}|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

Variational Principle

$$E_{trial} = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$$

Indistinguishable Particles

Bosons: spin $\in \mathbb{Z}$

$$\psi(x_1, x_2) = +\psi(x_2, x_1)$$

Fermions: spin $\in \mathbb{Z} + \frac{1}{2}$

$$\psi(x_1, x_2) = -\psi(x_2, x_1)$$

$$\psi_{sym.} = \frac{1}{\sqrt{2}}\left(\phi_a(x_1)\phi_b(x_2) + \phi_a(x_2)\phi_b(x_1)\right)$$

$$\psi_{antisym} = \frac{1}{\sqrt{2}}\left(\phi_a(x_1)\phi_b(x_2) - \phi_a(x_2)\phi_b(x_1)\right)$$

Electrons:

$$\chi_{sym} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$\chi_{antisym} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \quad (0)$$

$|\uparrow\uparrow\rangle$ (+1)

$|\downarrow\downarrow\rangle$ (-1)

Quantum #'s

n - energy lev.

ℓ - shape of orb. $\quad \ell \in \{0, \dots, n-1\}$

m_ℓ - orientation of orb. $\quad m_s \in \{+\frac{1}{2}, -\frac{1}{2}\}$

$\{-\ell, -\ell+1, \dots, \ell-1, \ell\}$

ℓ	0	1	2	3
	s	p	d	f

AMO

Madelung: Fill in order of $n+\ell$, same → smaller n first

Hund's:

1. Max $M = 2S+1$ ↔ lowest energy ↔ # unpaired e^- +1
2. For fixed M, max L ↔ min E
3. (i) $< \frac{1}{2}$ filled ↔ min $J = L+S$ ↔ min E
   (ii) $> \frac{1}{2}$ filled ↔ max $J = L+S$ ↔ min E

L-S Coupling:

$$L = (l_1 + l_2), (l_1 + l_2 - 1), \dots, (l_1 - l_2)$$

$$S = (s_1 + s_2), (s_1 + s_2 - 1), \dots, (s_1 - s_2)$$

$$J = (L+S), (L+S-1), \dots, (L-S)$$

Term Symbol: $\quad {}_n^{2S+1}L_J$

BO: $\frac{(b - ab)}{2}$

$$\psi_b = \psi_a + \psi_b$$

If hybridiz. $\psi_{ab} = \psi_a - \psi_b$

$$\pi < \sigma < \pi^* < \sigma^*$$

Dynamical Pictures

Schrödinger: $i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$

$$\langle A\rangle(t) = \langle\psi(t)|A|\psi(t)\rangle$$

Heisenberg:

$$A(t) = e^{iHt/\hbar}Ae^{-iHt/\hbar}$$

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[H, A(t)] + \left(\frac{\partial A}{\partial t}\right)_H$$

Dirac: $H = H_0 + V$ ← pert.

Operators evolve w/ H_0

States evolve w/ V

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle$$

Sudden Approx

$$|\psi(0^+)\rangle = \sum_m \langle m^{(1)}|n^{(0)}\rangle|m^{(1)}\rangle$$

if $|\psi(0^-)\rangle = |n^{(0)}\rangle$

WKB & B-S Quantization

$$\psi(x) = Ae^{\pm ikx} \quad k = \frac{\sqrt{2m(E-V)}}{\hbar} = \frac{p(x)}{\hbar}$$

Turning Pts: $E \approx V$

$$\psi(x) \approx \frac{c}{\sqrt{p(x)}}e^{\pm\frac{i}{\hbar}\int_{x_t}^{x}p(x)dx} \quad E > V(x)$$

$$\psi(x) \approx \frac{c}{|\sqrt{p(x)}|}e^{\pm\frac{1}{\hbar}\int_{x_t}^{x}|p(x)|dx} \quad E < V(x)$$

B-S Q. Cond. $\oint p(x)dx = (n + \frac{1}{2}\alpha + \frac{1}{4}\beta)$

α: # hard turning pts. $\quad \beta$: # soft turning pts.

Transmission Prob:

$$T = \exp\left(-\frac{2}{\hbar}\int_a^b |p(x)|^2 dx\right)$$

Dyson Expansion

$$i\hbar\frac{d}{dt}U_I(t, t_0) = V_I(t)U_I(t, t_0)$$

$$U_I(t, t_0) = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t}V_I(t')dt'\right)$$

$$U_I(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t}V_I(t_1)dt' - \frac{1}{\hbar^2}\int_{t_0}^{t}dt'\int_{t_0}^{t_1}dt'\,V_I(t_1)V_I(t_2) + \dots$$

TDPT

$$c_f^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}dt'\langle f|V(t')|i\rangle e^{i\omega_{fi}t'}$$

$$\omega_{fi} = (E_f - E_i)/\hbar$$

$$P_{i\to f} = |c_f^{(1)}(t)|^2$$

Fermi's Golden Rule

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\rho(E_f)$$

$\rho(E_f)$: density of states @ $E_f = E_i + \hbar\omega$

Adiabatic Approximation

$$|\psi(t)\rangle = e^{i\gamma_n(t)}e^{-\frac{i}{\hbar}\int_0^t E(t')dt'}|n(t)\rangle$$

$$\gamma_n(t) = i\int_0^t \langle n(t')|\frac{d}{dt'}|n(t')\rangle dt'$$

Scattering Theory

$$r\to\infty:\ \psi(\vec{r}) \approx e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 \quad \sigma_{tot} = \int d\Omega\frac{d\sigma}{d\Omega}$$

Born Approx: $V(\vec{r})$ weak

$$f(\vec{k}, \vec{k}') = -\frac{2m}{4\pi\hbar^2}\int d^3r\, e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}V(\vec{r})$$

Partial Wave Expansion: $V(r) = V(\vec{r})$

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty}R_\ell(r)P_\ell(\cos\theta)$$

$r\to\infty$: $u_\ell(r) \sim A\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)$

$$\psi(r,\theta) \approx \sum_\ell (2\ell+1)\frac{e^{i(kr-\ell\pi/2)} - S_\ell e^{-i(kr-\ell\pi/2)}}{2ikr}P_\ell(\cos\theta)$$

$$S_\ell = e^{2i\delta_\ell}$$

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\frac{e^{i\delta_\ell}\sin\delta_\ell}{k}P_\ell(\cos\theta)$$

Bloch's Thm.

$$V(\vec{r}+\vec{R}) = V(\vec{r}) = \alpha\,\delta(x-a)$$

$$\psi_{n\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}u_{n\vec{k}}(\vec{r})$$

k: crystal momentum; $u_{n\vec{k}}$: periodic in lattice

$$\psi(x+a) = e^{iqa}\psi(x);\ q = \frac{2\pi n}{Na}\ n\in\mathbb{Z}$$

Dirac Comb $\quad k = \frac{\sqrt{2mE}}{\hbar}$

$$z = ka,\ \beta = \frac{m\alpha a}{\hbar^2}$$

$$f(z) = \cos(z) + \beta\frac{\sin(z)}{z}$$

Zeeman Shift

$H' = -\vec{\mu}\cdot\vec{B}$

e^-: $\vec{\mu} = -g\mu_B \vec{J}/\hbar$

If $\vec{B} = B\hat{z}$ $\Delta E = g\mu_B m_j B$

$$g = 1 + \frac{j(j+1)+s(s+1)-l(l+1)}{2j(j+1)}$$

Split w/ $m_j \to m_l$

Stark Shift

$H' = -\vec{d}\cdot\vec{E}$; $\vec{d} = -e\vec{r}$

If $\vec{E} = E\hat{z} \to H' = -Eez$

Linear: degen states

$\Delta E = \langle nlm| -ezE |nl'm'\rangle$

Selection Rules: $\Delta l = \pm 1$ $\Delta m = 0$

Quadratic: Non-degen states

$$\Delta E^{(2)} = -\frac{1}{2}\alpha E^2;\ \alpha = 2\sum_{n\neq 0}\frac{|\langle n|ez|0\rangle|^2}{E_n - E_0}$$

Fine Structure

Typical H atom: $E_n = \frac{13.6\,eV}{n^2}$

- $\Delta E_{rel} \sim \frac{-1}{n^3}\left(\frac{Z\alpha}{n}\right)^4$
- $\Delta E_{so} \propto \frac{1}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)$

$$\to \Delta E_{fine} = \frac{E_n (Z\alpha)^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)$$

Hyperfine Structure

$\vec{F} = \vec{I} + \vec{J}$

$$\Delta_{hf} E = A\cdot\left(\frac{F(F+1) - I(I+1) - J(J+1)}{2}\right)$$

RWA

Terms like $\underbrace{e^{i(\omega_0-\omega)t} + e^{i(\omega_0+\omega)t}}$ non-resonant

Drop non-res @ $\omega \approx \omega_0$ & factors ≥ 1 (terms w/)

Eikonal Approximation

High Energy: $\psi(\vec{r}) \approx e^{ikz} e^{i\chi(\vec{b},z)}$

$$\chi(\vec{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V(\vec{b}, z')\,dz'$$

Multipole Expansion

$A(\vec{r}) = A_{monopole} + A_{dipole} + A_{quadropole} + \ldots$

Dipoles:

$E_1 = \vec{r}$; $M_1 = \vec{L} + 2\vec{S}$

Quad:

$E_2 = r^2 Y_{2m}(\theta,\phi)$

Each atomic orbital has parity $\pi_l = (-1)^l$

E: even $\leftrightarrow$ odd

M: even $\leftrightarrow$ even, odd $\leftrightarrow$ odd

Tricks & Useful Info

$\sigma_z|\pm\rangle = \pm|\pm\rangle$

$\sigma_y|\pm\rangle = \pm i|\mp\rangle$

$\sigma_x|\pm\rangle = |\mp\rangle$

If an op. commutes w/ all ops. in a CSCO, it is a func. of those ops.

H rot inv. about $\hat{n}$ $\Rightarrow [H, S_n] = 0$

$A|\psi\rangle = \alpha|\psi\rangle + \beta|\psi\rangle_\perp$

$\sigma_A = 0$ iff $|\psi\rangle$ is e-state

Commute $\Rightarrow$ common eigenbasis $\Rightarrow$ diagonal in same basis

Simplif. of Born Approx

$V(\vec{r}) = V(r)$

$$f(\theta) = -\frac{2\mu}{\hbar^2 q}\int_0^\infty r V(r)\sin(qr)\,dr$$

Short range:

$$f(\theta) \approx -\frac{2\mu}{\hbar^2}\int_0^R r^2 V(r)\,dr$$

~~Isotropic scattering~~

Math

$e^{i\theta} = \cos\theta + i\sin\theta$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\ln(1+x) = \sum_{n=1}(-1)^{n+1}\frac{x^n}{n}$$

$$\sin(x) = \sum_{n=0}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}(-1)^n\frac{x^{2n}}{(2n)!}$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-a(x+b)^2}dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{\infty} x e^{-x^2}dx = \frac{1}{2}$$

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx$$

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\psi}(k)e^{ikx}dk$$

$2\sigma_1\cdot\sigma_2 = \sigma^2 - \sigma_1^2 - \sigma_2^2$

Functions & Stuff

Legendre Polynomials:

$P_0(x) = 1$; $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$; $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Spherical Hankel Fns.

$h_l^{(1)}(kr) = j_l(kr) + i n_l(kr)$ out

$h_l^{(2)}(kr) = j_l(kr) - i n_l(kr)$ in

J can't go $0 \to 0$

Selection Rules:

E_1: $\Delta l = \pm 1$, $\Delta m = 0, \pm 1$

$\pi_i \neq \pi_f$

M_1: $\Delta l = 0$, $\Delta m = 0, \pm 1$

$\pi_i = \pi_f$

E_2: $\Delta l = 0, \pm 2$ $\pi_i = \pi_f$

$\Delta m = 0, \pm 1, \pm 2$

$$n_0(kr) = -\frac{\cos(kr)}{kr}$$

$$n_1(kr) = -\frac{\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{kr}$$

Spherical Bessel Fns.

$$j_0(kr) = \frac{\sin(kr)}{kr};\ j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}$$

$$j_2(kr) = \left(\frac{3}{(kr)^3} - \frac{1}{kr}\right)\sin(kr) - \frac{3\cos(kr)}{(kr)^2}$$

Trial Wavefunctions

Harmonic Osc. $e^{-\alpha x^2}$, $e^{-\alpha x^2}H_n(\sqrt{2\alpha}\,x)$

Coulomb $e^{-\alpha r}$, $r^n e^{-\alpha r}$, $(1+\beta r)e^{-\alpha r}$ (non-gnd state) (multi e-)

Inf. Sq. well $\cos(\alpha x)$ or $\sin(\alpha x)$, $\sin^n\left(\frac{\pi x}{L}\right)$

Periodic $e^{ikx}u_k(x)$

Finite Sq. well $\psi_{in}(x) = A\cos(kx)$ $\psi_{in}(x) = A\sin(kx)$

$\psi_{out}(x) = Be^{-\alpha|x|}$ $\psi_{out}(x) = B\,\mathrm{sgn}(x)e^{-\alpha|x|}$

Known Results

Inf. Sq. well $V(x) = 0$ $0 < x < a$; $V(x) = \infty$ else

$$\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)$$

$$E_n = \frac{\hbar^2\pi^2 n^2}{2ma^2}\quad n \geq 1$$

Fin. Sq. well $V(x) = \begin{cases} -V_0 & |x| < a \\ 0 & |x| \geq a\end{cases}$

$$\psi(x) = \begin{cases} Ge^{kx} & x < -a \\ A\cos(kx) + B\sin(kx) & |x| < a \\ Fe^{-kx} & x > a\end{cases}$$

$$E = \frac{\hbar k^2}{2m} - V_0$$

Coulomb $V(x) = \frac{-e^2}{4\pi\epsilon_0 r}$ $\psi_{nlm} = R_{nl}(r)Y_l^m(\theta,\phi)$

$$E_n = \frac{-\mu e^4}{2(4\pi\epsilon_0)^2\hbar^2 n^2}$$

Harmonic Osc $V(x) = \frac{m\omega^2 x^2}{2}$ $\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}$

$E_n = \hbar\omega(n + \frac{1}{2})$ $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$

Free Particle $V(x) = 0$ $\psi_k = \frac{1}{\sqrt{2\pi}}e^{ikx}$ $E = \frac{\hbar^2 k^2}{2m}$

Delta Fn. $V(x) = \lambda\delta(x-a)$ $\psi(x) = \begin{cases}\psi_L = \sqrt{\kappa}\,e^{\kappa x} & x \leq a \\ \psi_R = \sqrt{\kappa}\,e^{-\kappa x} & x > a\end{cases}$

$\kappa = -\frac{m\lambda}{\hbar^2}$ $E = -\frac{m\lambda^2}{2\hbar^2}$

Radial wavefuncs

$$R_{nl}(r) = \sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-l-1)!}{2n((n+l)!)}}\,e^{-r/na_0}\left(\frac{2r}{na_0}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)$$

Ass Laguerre Poly

$$L_n^k = \frac{x^{-k}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+k}\right)$$

δ-fn stuff

$$\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^{(3)}(\vec{r})$$

$$\nabla^2\left(\frac{1}{|\vec{r}|}\right) = -4\pi\delta^{(3)}(\vec{r})$$

$V(\vec{r}) = V_0 \delta^3(\vec{r})$

$$f(\theta) = \frac{-2\mu}{4\pi\hbar^2} V_0$$

Two-Body Scattering

$V = V(r_1 - r_2)$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \quad R = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$H = \frac{\hbar}{2M}\left(\nabla^2_{\vec{R}} - \nabla^2_{\vec{r}}\right) + V(\vec{r})$$

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\psi(R, r) = \psi_{com}(\vec{R})\,\psi_{rel}(\vec{r})$$

$$\frac{-\hbar^2}{2M}\nabla^2_R \psi_{com} = E_{com}\psi_{com} \text{ (free)}$$

$$\left[\frac{-\hbar}{2\mu}\nabla^2_r + V(\vec{r})\right]\psi_{rel} = E_{rel}\psi_{rel} \text{ (pot in } V\text{)}$$

Boundary Conditions

ψ cont.

ψ' cont

$\psi = 0$ @ hard pts.