PHYS 137A FINAL EXAM CRIB SHEET

DIVIT RAWAL

Postulates

(i) All info ab. a sys is in $|\psi\rangle$

(ii) A physical observable is a Hermitian ($A = A^\dagger$) matrix

(iii) Possible results of observable A are a_n s.t. $A|\psi\rangle = a_n|\psi\rangle$

(iv) $\text{Prob}(a_n) = |\langle a_n|\psi\rangle|^2$

(v) After measurement: $|\psi'\rangle = \dfrac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$ (norm proj. of original ket onto kets corr. to measurement)

(vi) Time Evolution:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$\langle a|b\rangle \doteq$ a given b

$\langle a| \doteq |a\rangle^\dagger$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

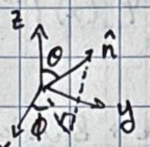
$$\det(A - \lambda I) = 0$$

Basis must satisfy:

- $|\psi\rangle = \sum_i \langle a_i|\psi\rangle |a_i\rangle$ (completeness)
- $\langle\psi|\psi\rangle = 1$ (normalization)
- $\langle a_i|a_j\rangle = \delta_{ij}$ (orthonormality)

Matrix Elements:

$$\langle\phi|A|\psi\rangle = \langle \overset{\text{row}}{n}|A|\overset{\text{col}}{m}\rangle$$



Expectations & Uncertainty:

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_n a_n \text{Prob}(a_n)$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

Matrix Rep. of Spins: $(\tfrac{1}{2})$

$|+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix}$ $|+\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ $|+\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$|-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$ $|-\rangle_x \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ $|-\rangle_y \doteq \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$S_z \doteq \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $S_x \doteq \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $S_y \doteq \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3\hbar^2}{4}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$S_n = \mathbf{S}\cdot\hat{n} = S_x \sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta$$

for

$$\hat{n} = \hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\Rightarrow S_{\hat{n}} = \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$$

$$\Rightarrow |+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$$

$$|-\rangle_n = \sin\frac{\theta}{2}|+\rangle - \cos\frac{\theta}{2}e^{i\phi}|-\rangle$$

$$[A, B] = AB - BA = -[B, A]$$

$[A, B] = 0 \Leftrightarrow A$ & B commute $\Leftrightarrow$ simoultaneously measureable $\Leftrightarrow$ share an eigenbasis

$\langle a|b\rangle = \langle b|a\rangle^\dagger$ $\quad \Delta A\Delta B \geq \frac{1}{2}|\langle[A, B]\rangle|$ $\quad \langle S^2\rangle = \frac{3\hbar}{4}$

Spin-1 Systems:

$S_z|1\rangle = \hbar|1\rangle$

$S_z|0\rangle = 0\hbar|0\rangle$

$S_z|-1\rangle = -\hbar|-1\rangle$

$$S_z \doteq \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$|1\rangle_x \doteq \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$

$|0\rangle_x \doteq \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|-1\rangle$

$|-1\rangle_x \doteq \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$

$$S_x \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \qquad S_y \doteq \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$$

For a spin s system, there are $2s+1$ states

$|1\rangle_y \doteq \frac{1}{2}|1\rangle + \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$

$|0\rangle_y \doteq \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle)$

$|-1\rangle_y \doteq \frac{1}{2}|1\rangle - \frac{i}{\sqrt{2}}|0\rangle - \frac{1}{2}|-1\rangle$

$$P_n = |\langle a_n|\psi\rangle|^2 = \langle a_n|\psi\rangle^*\langle a_n|\psi\rangle = \langle\psi|a_n\rangle\langle a_n|\psi\rangle = \langle\psi|P_n|\psi\rangle$$

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_n a_n P_n$$

$$[S_x, S_y] = i\hbar S_z$$
$$[S_y, S_z] = i\hbar S_x$$
$$[S_z, S_x] = i\hbar S_y$$

Schrödinger Time Evolution

$\cancel{H(t)}$ $H|E_n\rangle = E_n|E_n\rangle$ $\quad |\psi(t)\rangle = \sum_n c_n(t)|E_n\rangle$ s.t. $\langle E_k|E_n\rangle = \delta_{kn}$

$\Rightarrow |\psi(t)\rangle = \sum_n c_n(0)\, e^{-iE_n t/\hbar}|E_n\rangle$

H must be Unitary $(U^\dagger U = UU^\dagger = \mathbb{I})$

Time-Dependent Problems:

1. Diagonalize H (find e-vals. & e-vecs.)
2. Write $|\psi(0)\rangle$ in terms of $|E_n\rangle\ \forall n$
3. Multiply each e-state coeff by $e^{-iE_n t/\hbar}$ to get $|\psi(t)\rangle$
4. Calc. $\text{Prob}_{a_n} = |\langle a_j|\psi(t)\rangle|^2$

End of MT1 content

$\hat{x} = x \quad \hat{p} = -i\hbar\frac{\partial}{\partial x} \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad \hat{H}\varphi_{E_i}(x) = E_i\varphi_{E_i}(x) = \left(-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\varphi_{E_i}$

$\text{Prob}(x) = |\psi(x)|^2 \quad \text{Prob}(a<x<b) = \int_a^b |\psi(x)|^2\,dx \quad \text{Prob}(\psi\to\phi) = \int_{-\infty}^{\infty}\phi^*(x)\psi(x)\,dx$

$\langle\hat{x}\rangle = \int_{-\infty}^{\infty}\psi^*(x)\,x\,\psi(x)\,dx \quad \langle\hat{p}\rangle = \int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)\,dx$

Infinite Square Well:

$V(x) = \begin{cases} 0 & 0<x<L \\ \infty & \text{else}\end{cases} \quad \varphi_E = \begin{cases} A\sin kx + B\cos kx & 0<x<L \\ 0 & \text{else}\end{cases}$ for $k = \frac{2mE}{\hbar^2}$

Inside, $\frac{\partial^2}{\partial x^2}\varphi_E(x) = k^2\varphi_E(x) \quad \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ & $\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$ for $n\in\mathbb{Z}_{>0}$

Boundary Conditions: $\psi(x)$ continuous everywhere, $\frac{\partial\psi(x)}{\partial x}$ continuous except where $V(x)=\infty$

Finite Square Well:

$V(x) = \begin{cases} 0 & -a<x<a \\ V_0 & \text{else}\end{cases}$ Inside, $\frac{\partial^2}{\partial x^2}\varphi_E(x) = k^2\varphi_E(x)$ Outside, $\frac{\partial^2}{\partial x^2}\varphi_E(x) = q^2\varphi_E(x)$ for $q = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$

Break into even & odd states (cos & sin)

Exp. decay outside classically allowed region (tunneling)

Even: $\psi(x) = \psi(-x)$

Odd: $\psi(x) = -\psi(-x)$

Ehrenfest's Thm.: $\langle p(t)\rangle = m\frac{\partial\langle x(t)\rangle}{\partial t}$

$\int_{-a}^{a}\text{odd} = 0$

$\int_{-a}^{a}\text{even} = 2\int_0^a\text{even}$

Superposition & Time Dependence:

$\psi(x) = \sum_n c_n\varphi_n(x)$ for $c_n = \int_{-\infty}^{\infty}\varphi_n^*(x)\,\psi(x,0)\,dx$

Then, $\psi(x,t) = \sum_n c_n\, e^{-iE_n t/\hbar}\varphi_n(x)$

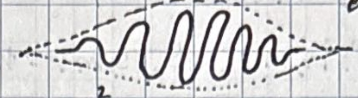
Unbound States:

$$\psi_E(x,t) = Ae^{ik(x-\omega t/k)} + Be^{-ik(x-\omega t/k)} \qquad |v| = \frac{\omega}{k}$$ phase velocity

$$\varphi_k(x) = Ae^{ikx} \quad \& \quad \hat{p}\,\varphi_k(x) = \hbar k\, \varphi_k(x) \Rightarrow p = \hbar k \Rightarrow \lambda_{debroglie} = \frac{h}{p}$$

$|v| = \frac{p}{2m}$ envelope velocity. Envelope moves w/ $\frac{1}{2}$ velocity of a given point inside. Position oscillates, but is localized to wavepackets

Position & Momentum are related by Fourier Transform:



Gaussian: $f(z) = \frac{e^{-(z-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$

$$\psi(x,t) = \sum_j c_j\, \varphi_{p_j}(x)\, e^{-iE_j t/\hbar} \text{ for } E_j = \frac{p_j^2}{2m} \qquad \langle x\rangle = \frac{p_0}{m}t$$

$$\phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x,t)\,e^{-ipx/\hbar}\,dx \qquad [\hat{x},\hat{p}] = i\hbar \Rightarrow \Delta x\Delta p \geq \frac{\hbar}{2}$$

Scattering:

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \qquad V(x) = \begin{cases} -V_0 & -a<x<a \\ 0 & \text{else} \end{cases} \qquad \psi_E = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x<a \\ Ce^{ik_2x} + De^{-ik_2x} & -a<x<a \\ Fe^{ik_1x} & x>a \end{cases}$$

$$k_2 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \qquad T = \frac{|F|^2}{|A|^2} \quad R = \frac{|B|^2}{|A|^2} \text{ s.t. } T+R=1$$

$E = -V_0 + \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$ when $2a$ contains an integer # of half wavelengths, $T=1$

T gives Prob(tunnell)

Quantum Harmonic Oscillator: $\omega = \sqrt{\frac{k}{m}}$ $\quad H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$

Raising Op: $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$ $\quad$ Lowering Op: $a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$ $\quad E_n = \hbar\omega\left(n+\frac{1}{2}\right)$ for $n \in \mathbb{Z}_{>0}$

Then,

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \qquad [a,a^\dagger] = 1,\ [H,a] = -\hbar\omega a,\ [H,a^\dagger] = \hbar\omega a^\dagger$$

$$H|E-\hbar\omega\rangle = (E-\hbar\omega)|E-\hbar\omega\rangle \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$a|E_{lowest}\rangle = 0$, $E_{lowest} = \frac{\hbar\omega}{2}$ $\qquad \varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ $\qquad$ H_n are Hermite polynomials & even

Let $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$ s.t. $\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}$

$$\varphi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)\,e^{-\xi^2/2}$$

Delta Potential:

$V(x) = \lambda\,\delta(x)$ (well if $\lambda<0$, barrier else)

Bound state ($E<0$): $E = -\frac{m\lambda^2}{2\hbar^2}$

Scattering ($E>0$): $T = \frac{1}{1+\frac{m\lambda^2}{2\hbar^2 E}}$, $R = \frac{1}{1+\frac{2\hbar^2 E}{m\lambda^2}}$

End of MT 2 content

Angular Momentum

$L = r \times p$ s.t. $L_x = yp_z - zp_y$ $\quad L_y = zp_x - xp_z$ $\quad L_z = xp_y - yp_x$

$[L_x, L_y] = i\hbar L_z$

$[L_y, L_z] = i\hbar L_x$ $\quad L^2 = L_x^2 + L_y^2 + L_z^2$

$[L_z, L_x] = i\hbar L_y$ $\quad [L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0$

Let $L_\pm = L_x \pm iL_y$ s.t. $[L_z, L_\pm] = \pm\hbar L_\pm$ & $[L^2, L_\pm] = 0$

$L_+|l, m\rangle = \hbar\sqrt{l(l+1) - m(m+1)}\,|l, m+1\rangle$ $\quad L^2 f_l^m = \hbar^2 l(l+1) f_l^m$ for $l = 0, \frac{1}{2}, 1, \frac{3}{2}$ etc.

$L_-|l, m\rangle = \hbar\sqrt{l(l+1) - m(m-1)}\,|l, m-1\rangle$ $\quad m = -l, -l+1, \dots, l-1, l$

$L_z f_l^m = \hbar m f_l^m$

$S^2|s, m\rangle = \hbar^2 s(s+1)|s, m\rangle$, $\quad S_z|s, m\rangle = \hbar m|s, m\rangle$

$S_\pm|s, m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\,|s, m\pm1\rangle$ for $S_\pm = S_x \pm iS_y$ $\quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ $\quad m = -s, -s+1, \dots, s-1, s$

$J = L + S$ $\quad j$ (total ang mom quantum #) $= |l-s|, |l-s|+1, \dots, l+s$ $\quad |J| = \hbar\sqrt{j(j+1)}$

Addition of Ang. Mom.

Combining spins s_1, s_2 gives $s = (s_1+s_2), (s_1+s_2-1), \dots, |s_1-s_2|$

$$|s\, m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1 s_2 m_1 m_2\rangle \quad \& \quad |s_1 s_2 m_1 m_2\rangle = \sum_s C^{s_1 s_2 s}_{m_1 m_2 m} |s\,m\rangle \quad (m = m_1 + m_2)$$

CG Table by $s_1 \times s_2$. Row by $m_1 \times m_2$. Col by $s\,m$

Hydrogen Atom

For 3-D, general $\hat{H} = -\frac{\hbar}{2m}\vec{\nabla}^2 + \hat{V}(\vec{r})$

Hydrogenic Atom V: $V(\vec{r}) = \frac{-Ze^2}{(4\pi\epsilon_0)r} \Rightarrow \hat{H} = \frac{p^2}{2\mu} - \frac{Ze^2}{(4\pi\epsilon_0)r}$; $\mu = \frac{mM}{m+M}$

degen n^2 $\rightarrow$ $E_n = -\frac{\mu e^4}{2\hbar^2(4\pi\epsilon_0)^2 n^2}$ for $n = 1, 2, \dots$

$n = 1, 2, 3, \dots$

$l = 0, 1, \dots, n-1$

$m = -l, \dots, l$

$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

Jacobian: $r^2\sin\theta$

$$\Psi_{nlm}(r, \theta, \phi) = R_{n,l}(r)\, Y_l^m(\theta, \phi)$$

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta, \phi)\, Y_{l'}^{m'*}(\theta, \phi) \sin\theta\, d\theta\, d\phi = \delta_{ll'}\delta_{mm'} \qquad \int_0^\infty R_{n,l}(r) R_{n',l}(r)\, r^2\, dr = \delta_{nn'}$$

For radial wave functions, # nodes $= n - l - 1$ $\quad$ Bohr Radius: $a = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}$

$E_n = -\frac{\hbar}{2m}\left(\frac{1}{an}\right)^2$ $\quad u_{nl} = rR_{n,l}(r) = \rho^{l+1} e^{-\rho} L^{2l+1}_{n-l-1}(2\rho)$

Dimensionless nth radius $= \rho = \frac{r}{an}$ ass.

$R_{n,l}(r) = e^{-r/na}\, r^l \sum_{j=0}^\infty c_j r^j$ $\quad R_{n,l}(r) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)}{2n[(n+l)!]}}\, e^{-\rho/2} \rho^l L^{2l+1}_{n-l-1}(\rho)$ $\leftarrow$ Laguerre polynomial

for $\rho = \frac{2r}{na}$

$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}$$

ass. Legendre polynomial