MATH 128A Crib Sheet - DIVIT RAWAL

Newton's Method: $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$ Secant: Newt w/ $f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{(p_{n-1}) - (p_{n-2})}$

Bisection: $ab<0$

FPI: $g(p)=p \rightarrow p_n = g(p_{n-1})$, $|g'(x)| \le 1$ **Conv.** $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|^{\alpha}} = \lambda$; α: ord, λ: rate

Lag. Interpol: $P = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$, $L_{n,k} = \prod_{i=0, i\ne k}^{n} \frac{(x-x_i)}{(x_k - x_i)}$

Div. Diff $f[x_k] = f(x_k)$; $f[x_k, \dots, x_j] = \frac{f[x_{k+1}, \dots, x_{k+j}] - f[x_k, \dots, x_{k+j-1}]}{x_{k+j} - x_k}$; Newt. DD: $P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, \dots, x_k](x-x_0) \cdots (x - x_{k-1})$

Cubic Splines: Interpol. pts, cont., S' cont, endpoints: ($S''(x) = 0$ (nat), $S'(x) = c$ (clamp)).

Taylor Exp $f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Richardson Extrap: Sup. $A(h) = A + c_1 h^p + c_2 h^{p+1} + \dots$; $A_{r,2h} = \frac{r^p A(h/r) - A(h)}{r^p - 1}$

Lipschitz: $\frac{|f(t,y_1) - f(t,y_2)|}{|y_1 - y_2|} \le L$ or $|\frac{\partial f}{\partial y}| \le L$

Simpson's: $O(h^4)$ $\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$; **Comp. Simps.** $O(h^4)$; $\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(x_0) + 4\sum_{\text{odd } i} f(a+ih) + 2\sum_{\text{even } i} f(a+ih) + f(x_n)\right)$; n even subintervals

D.o.P.: highest deg. poly s.t. approx is exact

Trap. Rule: $\int_a^b f(x)\,dx = \frac{h}{2}(f(a) + f(b))$

Convex: $(t_1, y_1), (t_2, y_2) \in D \Rightarrow \lambda(t_1, y_1) + (1-\lambda)(t_2, y_2) \in D$

IVP: $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$ well-posed if $f(t,y) \in C[(y),(t)]$ & Lipschitz in y

Euler's Method: $w_0 = \alpha$, $w_{i+1} = w_i + h f(t_i, w_i)$

HO. Taylor Methods: $w_0 = \alpha$, $w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$ ← $y(t)$, chain rule; order n, local trunc err $O(h^n)$

Mod. Euler: $w_0 = \alpha$, $w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right)$.

Systems of DiffEqs: $|f(t, u_1, \dots, u_m) - f(t, z_1, \dots, z_m)| \le L \sum_{j=1}^{m} |u_j - z_j|$

H.O. DiffEqs: write as a system of Diff Eqs.

Stability: Consistent: $\lim_{h\to 0} \max_{1\le i \le N} |\tau_i(h)| = 0$

Convergent: $\lim_{h \to 0} \max_{1 \le i \le N} |w_i - y(t_i)| = 0$

Sup. $w_0 = \alpha$, $w_{i+1} = w_i + h\phi(t_i, w_i, h)$ $\exists h_0 > 0$ s.t. ϕ cont. & Lips on D $\Rightarrow$ stable, conv. iff consistent; ie. conv = cons. + stab. (strong)

LMMs: $\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$. Def. $\rho(z) \doteq \sum_{j=0}^{k} \alpha_j z^j$; $\sigma(z) \doteq \sum_{j=0}^{k} \beta_j z^j$. **Zero-Stable** (strong) iff $\rho(z) = 0 \rightarrow |z| \le 1$, $|z|=1$ has multiplicity 1. **Weak Stable** if some root $|z|=1$ has multipl. >1

Absolute Stability: $y' = \lambda y$. $z \doteq h\lambda$; $y_{n+1} = R(z) y_n \rightarrow \rho(\xi) - z\sigma(\xi) = 0$ want $|\xi| < 1$

Diag. Dom $|a_{ii}| \ge \sum_{j \ne i} |a_{ij}| \rightarrow$ no piv. nec (if strict)

Gauss Elim Alg.
$\forall k = 1, \dots, n-1$: ← Fwd elim
(PIV) swap R_k w/ lower R w/ $\ge |a_{ik}|$
$\forall$ row $i = k+1, \dots, n$:
$m_{ik} \leftarrow \frac{a_{ik}}{a_{kk}}$
$R_i \leftarrow R_i - m_{ik} R_k$
$\forall i = n, n-1, \dots, 1$: ← Back sub
$x_i = \frac{1}{u_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j\right)$
$O(\frac{2}{3}n^3 + \frac{1}{2}n^2)$

LU Decomp. Alg
$U \leftarrow A$; $L \leftarrow I_n$
$\forall$ col $k = 1, \dots, n-1$
(PIV) $k \leftrightarrow p$ in both U & L (below diag)
$\forall R_i = k+1, \dots, n$
$L_{ik} \leftarrow l_{ik} = \frac{u_{ik}}{u_{kk}}$
$U_{i,*} \leftarrow U_{i,*} - l_{ik} U_{k,*}$
$O(n^3)$

Chol: A is SPD $\Rightarrow A = LL^T \rightarrow$ no piv. needed. (Chol works, faster than LU)

PD: $x^T A x > 0$ $\forall x \ne \vec{0} \rightarrow$ leading princ minors $> 0 \Rightarrow$ safe LU decomp

Det: $\det(AB) = \det(A)\det(B)$; $\det(A^T) = \det(A)$; $\det(A^{-1}) = \frac{1}{\det(A)}$; $\det(kA) = k^n \det(A)$; $\det(A) = 0 \Rightarrow$ sing. $\Rightarrow$ not invble

$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$ $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei + bfg + cdh - ceg - bdi - afh$

Pivots are diag(U) if L is unit-low-tri

Composite Trap: $\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right)$ $O(h^2)$; $x_j = a + jh$ Err: $-\frac{(b-a)h^2}{12} f''(\xi)$

Hermite Interpol. Repeat nodes w/ der. $f[x_i, x_i] = f'(x_i)$ use Newt. DD.

Banded Mats. LU decomp: $O(np^2)$ (p-band)

Pivoting: swap rows of P (starts as I_n) when swap rows of A

$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$ ← Leg. n

Ex. Derive RK-2 ← $O(h^3)$ ← second ord. acc

$y_{n+1} = y_n + h\Phi(f, t_n, y_n)$

$y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2} y''(t_n) + O(h^3)$ (*)

$y' = f$; $y''(t_n) = \frac{d}{dt} f(t,y) = f_t + f_y \cdot f$ (chain)

$k_1 = f(t_n, y_n)$; $k_2 = f(t_n + \alpha h, y_n + \beta h k_1)$;

$y_{n+1} = y_n + h(ak_1 + bk_2)$

T.E. $\rightarrow k_2 = f(t_n + \alpha h, y_n + \beta h f(t_n, y_n))$
$= f + \alpha h f_t + \beta h f_y f + O(h^2)$

Plug $k_i \rightarrow y_{n+1} = y_n + h(af + b(f + \alpha h f_t + \beta h f_y f)) + O(h^3)$
$= y_n + hf(a+b) + h^2 b(\alpha f_t + \beta f_y f) + O(h^3)$

From (*), hf: $a + b = 1$, h^2: $b\alpha = \frac{1}{2}$, $b\beta = \frac{1}{2}$

pick b, solve.

Ex. Taylor Matching $\tau(h) = y_{k+1} - w_{k+1}$

$w_{k+1} = -4w_k + 5w_{k-1} + h(\gamma f(t_k, w_k) + \beta f(t_{k-1}, w_{k-1}))$

γ, β s.t. order 2.

$\rightarrow w_{k+1} + 4w_k - 5w_{k-1} = h(\gamma f(t_k, w_k) + \beta f(t_{k-1}, w_{k-1}))$

$\alpha_2 = 1$, $\alpha_1 = 4$, $\alpha_0 = -5$; $\beta_2 = 0$, $\beta_1 = \gamma$, $\beta_0 = \beta$ in std.

$y(t_k) = y$, $y(t_{k+1}) = y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' \cdots$ form

$y(t_{k-1}) = y - hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' \cdots$

Plug in & eval $\rightarrow$

$y_{k+1} + 4y_k - 5y_{k-1}$
$= 0 + 6y'h - 2h^2 y'' + h^3 y''' \cdots$

$\rightarrow h(\gamma y' + \beta(y' - hy'' + \frac{h^2}{2}y''')) = h(\gamma + \beta)y'$
$- h^2 \beta y'' + \frac{h^3}{2} y''' \beta$

LHS - RHS $= \tau(h) = O(h^3) \Rightarrow 6 = \gamma + \beta$
$-2 = \beta$

solve.

Every iteration is FPI

Louse FPT to show conv
↳ map to self,
$|g'(x)| < 1$

as $k \to \infty$
$|x_k - x^*| \to 0$
if convgs.