PHYSICS 112 Crib Sheet DIVIT RAWAL

Ch1 $(x+y)^N = \sum_{t=0}^{N} \frac{N!}{(N-t)!\,t!} x^t y^{N-t}$

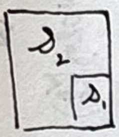
Binary system $= \sum_{s=-\frac{N}{2}}^{\frac{N}{2}} \frac{N!}{(\frac{N}{2}-s)!(\frac{N}{2}+s)!} x^{\frac{N}{2}-s} y^{\frac{N}{2}+s}$

$\lim s/N \ll 1,\ N \gg 1$ $g(N,s) = g(N,0)\exp(-2s^2/N)$

for $g(N,0) = \left(\frac{2}{\pi N}\right)^{1/2} 2^N$ $\langle f \rangle = \sum_s f(s)\Pr[s]$

$\sum_s g(N,s) = 2^N$ For N magnets in uniform field, $U = -2smB$

Ch2 Fundamental Ass. $\Pr[s] = 1/g$



$N = N_1 + N_2$ & N_i const.

$U = U_1 + U_2$ & U const

$dU_1 = -dU_2$

$g(N,U) = \sum_{U_1 \le U} g_1(N_1,U_1)\, g_2(N_2, U-U_1)$

Therm. Eq. $\Rightarrow dg = 0$

$\to \left(\frac{\partial g_1}{\partial U_1}\right)_{N_1} g_2\, dU_1 + \left(\frac{\partial g_2}{\partial U_2}\right)_{N_2} g_1\, dU_2 = 0$

$\to \frac{1}{g_1}\left(\frac{\partial g_1}{\partial U_1}\right)_{N_1} = \frac{1}{g_2}\left(\frac{\partial g_2}{\partial U_2}\right)_{N_2}$

$\to \left(\frac{\partial \log g_1}{\partial U_1}\right)_{N_1} = \left(\frac{\partial \log g_2}{\partial U_2}\right)_{N_2}$

$\langle E^2 \rangle = \frac{\partial^2}{\partial \beta^2} \log Z$

$\sigma \doteq \log g$ & $\frac{1}{\tau} \doteq \left(\frac{\partial \sigma}{\partial U}\right)_N$

$\tau = k_B T;\ S = k_B \sigma$

Consider $\Delta U > 0$ $\mathcal{S}_1 \to \mathcal{S}_2$

$\Delta\sigma = \left(\frac{\partial \sigma_1}{\partial U_1}\right)_{N_1}(-\Delta U) + \left(\frac{\partial \sigma_2}{\partial U_2}\right)_{N_2}(\Delta U)$

$= \Delta U\left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right)$

Laws of Thermodynamics

0. $\tau_1 = \tau_2,\ \tau_2 = \tau_3 \Rightarrow \tau_1 = \tau_3$
1. $U = \tau d\sigma$ (i.e. heat is a form of energy)
2. $\sigma_f \ge \sigma_i$ (law of inc of entropy)

For large N, n: $\frac{g(N,n)}{2^N} \approx \frac{1}{\sqrt{\pi\langle n\rangle}} \exp\left(-\frac{s^2}{\langle n\rangle}\right)$

Gaussian Distr.!! for $s \doteq n - \langle n \rangle$

Ch3. $\frac{\Pr[\varepsilon_1]}{\Pr[\varepsilon_2]} = \frac{g(U_0-\varepsilon_1)}{g(U_0-\varepsilon_2)}$

$= \exp\underbrace{(\sigma(U_0-\varepsilon_1) - \sigma(U_0-\varepsilon_2))}_{\Delta\sigma}$

Via Tayl. Exp. $\Delta\sigma = -(\varepsilon_1 - \varepsilon_2)/\tau$

$\to \frac{\Pr[\varepsilon_1]}{\Pr[\varepsilon_2]} = \frac{\exp(-\varepsilon_1/\tau)}{\exp(-\varepsilon_2/\tau)}$ $\quad Z \doteq \sum_s \exp(-\varepsilon_s/\tau)$

$\to \Pr[\varepsilon_s] = \exp(-\varepsilon_s/\tau)/Z$

$\Rightarrow U = \langle \varepsilon \rangle = \frac{\sum_s \varepsilon_s \exp(-\varepsilon_s/\tau)}{Z} = \tau^2 \frac{\partial \log(Z)}{\partial \tau} = \frac{\tau}{Z}\frac{\partial Z}{\partial \tau}$

$C_V \doteq \tau\left(\frac{\partial \sigma}{\partial \tau}\right)_V = \left(\frac{\partial U}{\partial \tau}\right)_V$

$C_p \doteq \tau\left(\frac{\partial \sigma}{\partial \tau}\right)_p = \left(\frac{\partial U}{\partial \tau}\right)_p$

Ass. Fact takes $V \to V - \Delta V$

$\varepsilon_s(V-\Delta V) = \varepsilon_s(V) - \frac{d\varepsilon_s}{dV}\Delta V + \dots$

$\to U(V-\Delta V) - U(V) = \Delta U = -\frac{d\varepsilon_s}{dV}\Delta V$

$\to$ Avg. pressure const. σ for reversibility $p = -\left(\frac{\partial U}{\partial V}\right)_\sigma$

$\to d\sigma(U,V) = \left(\frac{\partial \sigma}{\partial U}\right)_V dU + \left(\frac{\partial \sigma}{\partial V}\right)_U dV$

select dU, dV s.t.

$0 = \left(\frac{\partial \sigma}{\partial U}\right)_V (\delta U)_\sigma + \left(\frac{\partial U}{\partial V}\right)_U (\delta V)_\sigma$

$\to 0 = \left(\frac{\partial \sigma}{\partial U}\right)_V \left(\frac{\delta U}{\delta V}\right)_\sigma + \left(\frac{\partial U}{\partial V}\right)_U$

$\Rightarrow \left(\frac{\partial U}{\partial V}\right)_\sigma = -\tau\left(\frac{\partial \sigma}{\partial V}\right)_U = -p$

Thermodynamic Identity

$d\sigma = \frac{1}{\tau}dU + \frac{p}{\tau}dV$

$\Rightarrow \tau d\sigma = dU + pdV$

$dU = \underbrace{\tau d\sigma}_{\text{heat}} - \underbrace{pdV}_{\text{work}}$

$F \doteq U - \tau d\sigma$ @ const V,

$\frac{1}{\tau} = \frac{\partial \sigma}{\partial U} \Rightarrow dF = 0$ s.t.

F is min. @ const. V, τ

$dF = dU - \tau d\sigma - \sigma d\tau$

$= -\sigma d\tau - pdV$

$-\sigma = \left(\frac{\partial F}{\partial \tau}\right)_V;\ -p = \left(\frac{\partial F}{\partial V}\right)_\tau$

$\Rightarrow p = -\left(\frac{\partial U}{\partial V}\right)_\tau + \tau\left(\frac{\partial \sigma}{\partial V}\right)_\tau$

Maxwell Relns.

$\left(\frac{\partial \sigma}{\partial V}\right)_\tau = \left(\frac{\partial p}{\partial \tau}\right)_V$

$\left(\frac{\partial \tau}{\partial V}\right)_\sigma = -\left(\frac{\partial p}{\partial \sigma}\right)_V$

$F = -\tau \log Z \equiv Z = \exp(-F/\tau)$

$\to \Pr[\varepsilon_s] = \exp\left(\frac{F - \varepsilon_s}{\tau}\right)$

Ideal Gas: $PV = N\tau$

$Z = \frac{V}{\lambda^3}$ for $\lambda \doteq \sqrt{\frac{h^2}{2\pi m\tau}}$

$U = \frac{3}{2}\tau;\ \varepsilon_n = \frac{\hbar^2}{2M}\left(\frac{\pi}{L}\right)^2 (n_x^2+n_y^2+n_z^2)$

$n^2 \doteq n_x^2 + n_y^2 + n_z^2$

$Z_1 = \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \exp(-\alpha^2 n^2)$

$\alpha^2 \doteq \frac{\hbar^2\pi^2}{2ML^2\tau} \Rightarrow Z_1 = \pi^{3/2}/8\alpha^3$

$\to Z_1 \doteq \frac{n_Q}{n} = n_Q V;$ $\quad n \doteq \frac{1}{V};\ n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$

$1/8$ factor b/c pos. octant.

Factor of 2 for polarization

$\hbar = \frac{h}{2\pi}$

$Z_N = \frac{1}{N!} Z_1^N = \frac{1}{N!}(n_Q V)^N$

$\sigma \doteq N(\log(n_Q/n) + \frac{5}{2})$

$n \doteq N/V$ Sackur-Tetrode Eq.

Density of State

$g(E) = \frac{d\Omega}{dE}$ # q. states that become available $E \to E + dE$

$D(\varepsilon)d\varepsilon = \frac{A 2\pi p dp}{h^2}$ (2D)

$D(\varepsilon)d\varepsilon = \frac{V 4\pi p^2 dp}{h^3}$ (3D)

$E = ck^\alpha$ & dim d

$D_d(E) = g_s \frac{2^{1-d}}{\alpha c^{d/\alpha}} \frac{\pi^{-d/2}}{\Gamma(d/2)} E^{d/\alpha - 1}$

$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Find $p(\varepsilon)$, then diff.

Equipartition Thm: $\langle E \rangle = \frac{N\tau}{2}$ in each d.o.f.

Ch4 $\varepsilon_s = s\hbar\omega$ $\quad s \in \mathbb{Z}_{\ge 0}$

$Z = \sum_{s=0}^{\infty} \exp(-s\hbar\omega/\tau)$ $\quad x \doteq \exp(-\hbar\omega/\tau)$

$\to Z = \frac{1}{1 - \exp(-\hbar\omega/\tau)}$

$\to \langle s \rangle = \frac{1}{\exp(\hbar\omega/\tau) - 1}$ ← avg. # phonons in the mode

$\langle \varepsilon \rangle = \langle s \rangle \hbar\omega$ $\quad$ classical lim: $\tau \gg \hbar\omega$ $\langle \varepsilon \rangle \approx \tau$

Rad. in cube of length L,

$E_x = E_{x_0}\sin(\omega t)\cos(n_x\pi x/L)\sin(n_y\pi y/L)\sin(n_z\pi z/L)$

$E_y = E_{y_0}\sin(\omega t)\sin(n_x\pi x/L)\cos(n_y\pi y/L)\sin(n_z\pi z/L)$

$E_z = E_{z_0}\sin(\omega t)\sin(n_x\pi x/L)\sin(n_y\pi y/L)\cos(n_z\pi z/L)$

$\operatorname{div} E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \equiv \vec{E}_0 \cdot \vec{n} = 0$

$n^2 \doteq n_x^2 + n_y^2 + n_z^2$ $\quad \omega_n = \frac{n\pi c}{L}$

$U = \sum_n \langle \varepsilon_n \rangle = \left(\frac{\pi^2\hbar c}{L}\right)\left(\frac{\tau L}{\pi\hbar c}\right)^4 \int_0^\infty dx \frac{x^3}{\exp x - 1}$

$x \doteq \pi\hbar c n/L\tau$

$\frac{U}{V} = \frac{\pi^2}{15\hbar^3 c^3}\tau^4$ $\quad V = L^3$ ← Stefan-Boltzmann Law of rad.

Spectral density of radiation

u_ω ← energy / unit vol / unit freq. range

$\frac{U}{V} = \int d\omega\, u_\omega \Rightarrow u_\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}$

Planck → radiation law

$d\sigma = \frac{4\pi^2 V}{15\hbar^3 c^3}\tau^2 d\tau \Rightarrow \sigma(\tau) = (4\pi^2 V/45)(\tau/\hbar c)^3$

const. phonon $\sigma \Rightarrow V\tau^3$ const.

energy flux density J_u ← rate of E emission / unit area

$J_u = \sigma_B T^4$

$\sigma_B \doteq \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$ ← Stefan-Boltzmann const.

$a(\omega) = e(\omega)$

$a = e$ ← obj must absorb & emit @ same rate @ eq.
← Kirchoff Law

Perfect reflector does not radiate $a = e = 0$

Est. Surface Temp.

$\frac{d}{dx} = \frac{x^3}{\exp x - 1} = 0$

$\rightarrow \frac{\hbar\omega_{max}}{k_B T} = x_{max} \approx 2.82$

Debeye Theory

$\langle s(\omega)\rangle = \frac{1}{\exp(\hbar\omega/\tau)-1}$

$\frac{3}{8}\int_0^{n_{max}} 4\pi n^2 dn = 3N$

$n_D \equiv n_{max} \rightarrow \frac{1}{2}\pi n_D^3 = 3N$

$\rightarrow n_D \doteq \left(\frac{6N}{\pi}\right)^{1/3}$

$U = \sum \frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau)-1}$

$= \frac{3\pi}{2}\int_0^{n_D} dn\, n^2 \frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau)-1}$

$x \doteq \frac{\pi\hbar v}{L\tau}$ v ≡ velocity of sound

$\Rightarrow U = \left(\frac{3\pi^2\hbar v}{2L}\right)\left(\tau L/\pi\hbar v\right)^4 \int_0^{x_D} dx \frac{x^3}{\exp x - 1}$

$\rightarrow x_D = \frac{\pi\hbar v n_D}{L\tau} = \hbar v (6\pi^2 N/V)^{1/3}/\tau$

$= \theta/T = \frac{\theta k_B}{T}$ $\theta \doteq \left(\frac{\hbar v}{k_B}\right)\left(\frac{6\pi^2 N}{V}\right)^{1/3}$

$T \ll \theta$, $x_D \rightarrow \infty \Rightarrow U(T) = \frac{3\pi^4 N k_B T^4}{5\theta^3}$

$C_V = \frac{12\pi^4 N}{5}\left(\frac{\tau}{k_B\theta}\right)^3$

$= \frac{12\pi^4 N k_B}{5}\left(\frac{T}{\theta}\right)^3$

Ch5

$\mu(\tau, V, N) \doteq \left(\frac{\partial F}{\partial N}\right)_{\tau, V}$

Diffusive Eq: $\mu_1 = \mu_2$

Ideal Gas: $\mu = \tau\log(n/n_Q)$

$= \tau\log(N/Z_1) = \tau\log(p/\tau n_Q)$

$n = \frac{N}{V}$ as def. earlier

δ_1 $\boxed{\mu_1, \tau}$ $+\Delta V$ $\boxed{\mu_2, \tau}$ δ_2

$q\Delta V = \Delta\mu_i$

$\mu_{1f} = \mu_{1i} + (\mu_{2i} - \mu_{1i})$

$= \mu_{2i} = \mu_{2f}$

$\mu \doteq \mu_{tot} = \mu_{int} + \mu_{ext}$

$\mu_2 = \mu_1 \Rightarrow \Delta\mu_{ext} = -\Delta\mu_{int}$

$\frac{\mu(U,V,N)}{\tau} = -\left(\frac{\partial\sigma}{\partial N}\right)_{U,V}$

Thermodynamic Identity v2

$dU = \tau d\sigma - p dV + \mu dN$

$\boxed{R}$ ⓢ $\frac{\Pr[N_1, \varepsilon_1]}{\Pr[N_2, \varepsilon_2]} = \frac{g(N_0 - N_1, U_0 - \varepsilon_1)}{g(N_0 - N_2, U_0 - \varepsilon_2)} = \exp(\Delta\sigma)$

$g(N_0, U_0) = \exp(\sigma(N_0, U_0))$

$\sigma(N_0 - N, U_0 - \varepsilon) = \sigma(N_0, U_0) - N\left(\frac{\partial\sigma}{\partial N_0}\right)_{U_0} - \varepsilon\left(\frac{\partial\sigma}{\partial U_0}\right)_{N_0} + \ldots$

$\Rightarrow \Delta\sigma = \frac{(N_1 - N_2)\mu}{\tau} - \frac{(\varepsilon_1 - \varepsilon_2)}{\tau}$

$\rightarrow \mathcal{Z} \doteq \sum_{N=0}^{\infty} \lambda^N \sum_{S(N)} \exp((N\mu - \varepsilon_{S(N)})/\tau)$

$= \sum_{ASN} \exp(N\mu - \varepsilon_{S(N)}/\tau)$

$\Pr[N_1, \varepsilon_1] = \frac{\exp((N_1\mu - \varepsilon_1)/\tau)}{\mathcal{Z}}$

$\langle N\rangle = \frac{\tau}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial\mu} = \tau\frac{\partial\log\mathcal{Z}}{\partial\mu}$

$\lambda \doteq \exp(\mu/\tau)$ ← abs. activity

$\langle N\rangle = \lambda\frac{\partial}{\partial\lambda}\log\mathcal{Z}$

$U = \left(\frac{\mu}{\beta}\frac{\partial}{\partial\mu} - \frac{\partial}{\partial\beta}\right)\log\mathcal{Z}$

$= \left(\tau\mu\frac{\partial}{\partial\mu} - \frac{\partial}{\partial(1/\tau)}\right)\log\mathcal{Z}$

Ch6 Ideal Gas

Fermion: spin $\in \mathbb{Z}_{\geq 0} + \frac{1}{2}$

Boson: spin $\in \mathbb{Z}_{\geq 0}$

Orbital can be occupied by $n \in \mathbb{Z}_{\geq 0}$ bosons, $n \in \{0, 1\}$ fermions (Pauli Excl Princ.)

$f(\varepsilon, \tau, \mu)$ ← dist. fn.

$f \ll 1$ ← classical regime

Fermi-Dirac Dist. Fn.

$f(\varepsilon) \doteq \langle N(\varepsilon)\rangle = \frac{1}{\lambda^{-1}\exp(\varepsilon/\tau)+1}$

$\mathcal{Z} = 1 + \lambda\exp(-\varepsilon/\tau)$

$f(\varepsilon) = \frac{1}{\exp((\varepsilon-\mu)/\tau)+1} \in [0, 1]$

Fermi energy $\varepsilon_F = \mu(0)$

Bose-Einstein Dist. Fn.

$f(\varepsilon) = \frac{1}{\exp((\varepsilon-\mu)/\tau)-1}$

Classical Lim

avg. occ. of every orbital $\ll 1$

$\exp((\varepsilon - \mu)/\tau) \gg 1$

$\rightarrow f(\varepsilon) \approx \lambda\exp(-\varepsilon/\tau)$

$\lambda = \exp(\mu/\tau) = \frac{n}{n_Q}$

$\mu = \tau\log(n/n_Q)$

$F = N\tau(\log(\frac{n}{n_Q}) - 1)$

$pV = N\tau$, $U = \frac{3}{2}N\tau$

$\sigma = N(\log(n_Q/n) + \frac{5}{2})$

$C_V = \frac{3}{2}N$

$C_p = \tau\left(\frac{\partial\sigma}{\partial\tau}\right)_p = \left(\frac{\partial U}{\partial\tau}\right)_p + p\left(\frac{\partial V}{\partial\tau}\right)_p$

$\rightarrow C_p = C_V + N$ → R

$\equiv C_p = C_V + Nk_B$

$C_p = \frac{5}{2}N$

$C_p/C_V = \gamma = \frac{5}{3}$ for monatomic I.G.

Polyatomic Ideal Gas

$\varepsilon = \varepsilon_n + \varepsilon_{int}$

ε_n ← COM motion, ε_{int} ← rot & vib. d.o.f.

$\mathcal{Z} = 1 + \lambda Z_{int}\exp(-\varepsilon_n/\tau)$

for $Z_{int} \doteq \sum_{int}\exp(-\varepsilon_{int}/\tau)$

$\lambda = n/n_Q Z_{int}$

$\Rightarrow \mu = \tau\log\left(\frac{n}{n_Q}\right) - \log Z_{int}$

int. $F_{int} = -N\tau\log Z_{int}$

int. $\sigma_{int} \doteq -\left(\frac{\partial F_{int}}{\partial\tau}\right)_V$

$U = \frac{3}{2}N\tau$ only for trans. energy

isothermal: $\sigma(V) = N\log V + \text{const.}$

$W_{\Delta V} = -\int_{V_1}^{V_2} p\,dV = N\tau\log\left(\frac{V_2}{V_1}\right)$

Isentropic

$\tau_1 V_1^{\gamma-1} = \tau_2 V_2^{\gamma-1}$

$\tau_1^{\gamma/(1-\gamma)} p_1 = \tau_2^{\gamma/(1-\gamma)} p_2$

$p_1 V_1^\gamma = p_2 V_2^\gamma$

	rev. isothermal exp.	rev. isentropic exp.	irrev. exp. into vac.
$U_2 - U_1$	0	$\left(\left(\frac{V_1}{V_2}\right)^{2/3}-1\right)\cdot\frac{3}{2}N\tau$	0
$\sigma_2 - \sigma_1$	$N\log\left(\frac{V_2}{V_1}\right)$	0	$N\log\left(\frac{V_2}{V_1}\right)$
W	$-N\tau\log\left(\frac{V_2}{V_1}\right)$	$\left(\left(\frac{V_1}{V_2}\right)^{2/3}-1\right)\frac{3}{2}N\tau$	0
Q	$N\tau\log\frac{V_2}{V_1}$	0	0

Ch7 Fermi & Bose Gasses

Classical: $n \ll n_Q$

Quantum: $n \geq n_Q$

Quantum if $\tau < \tau_0 \doteq \left(\frac{2\pi\hbar^2}{M}\right) n^{2/3}$

Degen Gas: $\tau \ll \tau_0$

$f(\varepsilon) \approx \left(\frac{n}{n_Q}\right)\exp(-\varepsilon/\tau)$ for $\varepsilon_0 = 0$

Fermi Gas

Degen if $\tau \ll \varepsilon_F$

ε_F: energy of highest filled orbital @ $\tau = 0$

Hold N electrons

$\Rightarrow N = 2 \times \frac{1}{8} \times \frac{4\pi}{3} n_F^2$

$\Rightarrow n_F = \left(\frac{3N}{\pi}\right)^{1/3}$

Fermi temp $\tau_F = \varepsilon_{F_{n_F}}$

$U_0 = 2\sum_{n \leq n_F} \varepsilon_n = 2 \times \frac{1}{8} \times 4\pi \int_0 dn\, n^2 \varepsilon_n$

$= \frac{3}{5}N\varepsilon_F$; $P_0 = \frac{2}{5}\frac{U}{V}$

$\langle X\rangle = \int d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon, \tau, \mu) X(\varepsilon)$

$\mathcal{D}(\varepsilon) \doteq \frac{dN}{d\varepsilon} = \frac{3N(\varepsilon)}{2\varepsilon}$

$\mathcal{D}(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}$ ← 4π not 2π for bosons

$N = \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon, \tau, \mu)$

$U = \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon)\varepsilon f(\varepsilon, \tau, \mu)$

$N_0 = \int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon, \tau, \mu)$

$U_0 = \int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon)\varepsilon f(\varepsilon, \tau, \mu)$

Heat Cap. of e^- Gas

N e^- to $\tau = 0 \to \tau$

$\Delta U = U(\tau) - U(0)$

$$\Delta U = \int_0^\infty d\varepsilon\, \varepsilon D(\varepsilon) f(\varepsilon) - \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon D(\varepsilon)$$

$$\left(\int_0^{\varepsilon_F} + \int_{\varepsilon_F}^\infty\right) d\varepsilon\, \varepsilon_F f(\varepsilon) D(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon_F D(\varepsilon)$$

$$\to \Delta U = \underbrace{\int_{\varepsilon_F}^\infty d\varepsilon (\varepsilon - \varepsilon_F) f(\varepsilon) D(\varepsilon)}_{E \text{ for } \varepsilon_F \to \varepsilon > \varepsilon_F} + \underbrace{\int_0^{\varepsilon_F} d\varepsilon (\varepsilon_F - \varepsilon)(1 - f(\varepsilon)) D(\varepsilon)}_{E \text{ for } \varepsilon < \varepsilon_F \to \varepsilon_F}$$

$$C_{el} = \frac{dU}{d\tau} = \int_0^\infty d\varepsilon (\varepsilon - \varepsilon_F) \frac{df}{d\tau} D(\varepsilon)$$

$$\int_{-\infty}^\infty dx\, x^2 \frac{e^x}{(e^x+1)^2} = \frac{\pi^2}{3}$$

$\to \tau \ll \tau_F$, $C_{el} = \frac{1}{3}\pi^2 D(\varepsilon_F)\tau$

$\tau \gg \tau_F$, $C_{el} = \frac{1}{2}\pi^2 N \tau / \tau_F$

In metals, $\to C_V/\tau = \gamma + A\tau^2$

Boson Gas & Einstein Condensation

@ $\tau \approx 0 \to N = -\tau/\mu \to \mu = -\tau/N$

$\to \lambda \approx 1 - \frac{1}{N}$

For Spin-0: $D(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$

$N = \sum_n f_n = \underbrace{N_0(\tau)}_{\text{cond. phase}} + \underbrace{N_e(\tau)}_{\text{norm. phase}}$

$= N_0(\tau) + \int_0^\infty d\varepsilon\, D(\varepsilon) f(\varepsilon, \tau)$

Einstein Condensation Temp.

$$\tau_E \doteq \frac{2\pi\hbar^2}{M}\left(\frac{N}{2.612V}\right)^{2/3}$$

$$\to \frac{N_e}{N} \approx \left(\frac{\tau}{\tau_E}\right)^{3/2}$$

Ch8 Heat & Work

Heat: energy transfer to a sys. via therm. contact w/ reservoir

work: energy transfer to a sys. via change in ext. parameters that desc. a sys.

$d\sigma = dU/\tau$ for heat

work: $d\sigma = 0$

$đQ \doteq \tau d\sigma$ heat recieved during reversible process

$dU = đW + đQ \to đW = dU - \tau\sigma$

Carnot Ineq.

- All types of work are freely convertible
- Work can be completely conv. into heat, but not the inverse.
  - ↳ follows from unlim. entropy accumulation arg.

Transfer of heat & work b/w states are path dep.

Heat Engine

$Q_h \leftarrow$ input heat

@ τ_h, $\sigma_h = \frac{Q_h}{\tau_h}$

$Q_l \leftarrow$ heat leaving sys @ τ_l $\sigma_l = \frac{Q_l}{\tau_l}$

Reversible $\Rightarrow \sigma_l = \sigma_h$

$\Rightarrow Q_l = \left(\frac{\tau_l}{\tau_h}\right) Q_h$

$W = Q_h - Q_l$

$$\eta_C \doteq \left(\frac{W}{Q_h}\right)_{rev} = \frac{\tau_h - \tau_l}{\tau_h}$$

Ineqs. $\sigma_l \geq \sigma_h$

$Q_l \geq Q_h(\tau_l/\tau_h)$

$W \leq \eta_C Q_h$

$\eta = \frac{W}{Q_h} \leq 1 - \left(\frac{\tau_l}{\tau_h}\right) = \eta_C$

Refrigerators $\boxed{+}$

Consume work $\tau_l \to \tau_h$

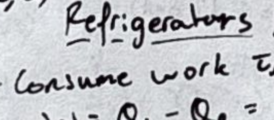
$W = Q_h - Q_l = \frac{\tau_h - \tau_l}{\tau_l} Q_l$

$\gamma_C = \left(\frac{Q_l}{W}\right)_{rev} = \frac{\tau_l}{\tau_h - \tau_l}$

$\gamma = \frac{Q_l}{W} \leq \gamma_C$

Carnot Cycle

$W = (\tau_h - \tau_l)(\sigma_h - \sigma_l)$



$1 \to 2$, $2 \to 3$ exp

$3 \to 4$, $4 \to 1$ compression

isoth. $1 \to 2$, $3 \to 4$

isentr. $2 \to 3$, $4 \to 1$

- All rev. energy conv. devices that op. b/w same temps have same $\eta = \frac{W}{Q_h}$

Irrev. if new entropy created

$đQ_{irrev} < đQ_{rev.}$

Isothermal Work

$đW = dF$

Isobaric - const. p

$đW' \doteq đW - d(pV)$

$= dH - đQ$

$H \doteq U + pV$

↳ enthalpy

$G = F + pV = U + pV - \tau\sigma$

↳ Gibbs Free Energy

$dU = \tau d\sigma - pdV + \mu dN$

$\to đW = \underbrace{-pdV}_{\text{mech. work}} + \underbrace{\mu dN}_{\text{chem. work}}$

Magnetic Work

$\frac{W_{mag}}{V} = \frac{B^2}{2\mu_0}$

$\delta W_{mag} = BdM$ (in sys?)

$\delta W_{mag} = -MdB$ (out sys?)

$$\frac{F_N(\tau) - F_S(\tau)}{V} = \frac{B_c^2(\tau)}{8\pi}$$

Ch9 Gibbs FE & Chem. Reactions

G @ min for sys. in therm eq. at const. p in therm contact w/ reservoir

$\to dG = 0$; $dG = \mu dN - \sigma d\tau + Vdp$

$\left(\frac{\partial G}{\partial N}\right)_{\tau,p} = \mu$; $\left(\frac{\partial G}{\partial \tau}\right)_{N,p} = -\sigma$

$\left(\frac{\partial G}{\partial p}\right)_{N,\tau} = V$; τ, p intensive

$G(N, p, \tau) = N\mu(p, \tau)$

$G = \sum_j N_j \mu_j$

$\sum \nu_j A_j = 0$

$dN_j = \nu_j d\hat{N}$ ← times reaction occurs.

$dG = \left(\sum_j \nu_j \mu_j\right) d\hat{N}$

↳ $= 0$ @ eq $\Rightarrow \sum_j \nu_j \mu_j = 0$

Eq. for Ideal Gasses

$\mu_j = \tau(\log n_j - \log c_j)$

$c_j \doteq n_{Qj} Z_j(\text{int})$

↳ dep. on temp but not concentration

$\sum_j \nu_j \log n_j = \sum_j \nu_j \log c_j$

$\Rightarrow K(\tau) \doteq \prod_j n_{Qj}^{\nu_j} \exp(-\nu_j F_j(\text{int})/\tau)$

↳ eq. const.

$K(\tau) = \prod_j n_j^{\nu_j}$ ← Law of Mass action

Ch10 Phase Transformations!

Isotherm: p vs. V @ const τ

Phase: portion of system uniform in composition

Coexistence Req. $\tau_1 = \tau_2$, $\mu_1 = \mu_2$, $p_1 = p_2$

$\mu_l(p,\tau) = \mu_g(p,\tau)$ lower is stable if not equal

Deriv. of Coexistence Curve

$\mu_g(p_0, \tau_0) = \mu_l(p_0, \tau_0)$ and

ass. $\mu_g(p_0 + dp, \tau_0 + d\tau) = \mu_l(p_0 + dp, \tau_0 + d\tau)$

Taylor Exp.

$$\to \frac{dp}{d\tau} = \frac{\left(\frac{\partial \mu_l}{\partial \tau}\right)_p - \left(\frac{\partial \mu_g}{\partial \tau}\right)_p}{\left(\frac{\partial \mu_g}{\partial p}\right)_\tau - \left(\frac{\partial \mu_l}{\partial p}\right)_\tau}$$

Def. $v \doteq \frac{V}{N}$, $s \doteq \sigma/N$

$\to \frac{dp}{d\tau} = \frac{s_g - s_l}{v_g - v_l}$

$L \doteq \tau(s_g - s_l)$ ← latent heat of vaporiz.

$\Delta v \doteq v_g - v_l$ $\frac{dp}{d\tau} = \frac{L}{\tau \Delta v}$

Clausius Clapeyron Eq./Vapor Pressure Eq.

if $v_g \gg v_l$, assume ideal gas law applies, $\Delta v \approx \tau/p$

$\to \frac{dp}{d\tau} = \frac{L}{\tau^2} p$, $\frac{d}{d\tau}\log p = \frac{L}{\tau^2}$

$\int \frac{dp}{p} = L_0 \int \frac{d\tau}{\tau^2}$

↳ if $L(\tau) = L_0$ $\forall\tau$ we want

$\to p(\tau) = p_0 \exp(-L_0/\tau)$

L_0 for single molecule, for a mole,

$p(T) = p_0 \exp(-L_0/RT)$

$R \doteq N_0 k_B$ ← Avogadro N_0 is const.

Triple Pt: $\mu_g = \mu_l = \mu_s$

Cross Coex. Curve:

$\tau d\sigma = dU + pdV - (\mu_g - \mu_l)dN$

$\mu_g = \mu_l \Rightarrow L = \tau\Delta\sigma = \Delta U + p\Delta V$

$= \Delta H = H_g - H_l$

$C_p = \left(\frac{\partial H}{\partial \tau}\right)_p \rightarrow H = \int C_p\, d\tau$

Vander Waals Eq. of State

$(p + N^2 a/V^2)(V - Nb) = N\tau$

a: long range attr.

b: short range repul.

F(ideal gas) $= -N\tau(\log(n_Q/n)+1)$

→ short hard core repul.

$V \to V - Nb$

$\to F = -N\tau(\log(n_Q(V-Nb)/N)+1)$

$P_c = \frac{a}{27b^2};\ V_c = 3Nb;\ \tau_c = \frac{8a}{27b}$

$\to \left(\frac{P}{P_c} + \frac{3}{(V/V_c)^2}\right)\left(\frac{V}{V_c} - \frac{1}{3}\right) = \frac{8\tau}{3\tau_c}$

$\hat p = P/P_c;\ \hat V = V/V_c;\ \hat\tau = \tau/\tau_c$

$\left(\hat p + \frac{3}{\hat V^2}\right)\left(\hat V - \frac{1}{3}\right) = \frac{8}{3}\hat\tau$

Law of corr. states

Critical Pt. $\left(\frac{\partial \hat p}{\partial \hat V}\right)_{\hat\tau} = 0;$

$\left(\frac{\partial^2 \hat p}{\partial \hat V^2}\right)_{\hat\tau} = 0$ if $\hat p = 1, \hat v = 1, \hat\tau = 1$

Above $\hat\tau_c$, no phase sep. exists.

$G(\tau, V, N) = \frac{N\tau V}{V - Nb} - \frac{2N^2 a}{V} - N\tau(\log(n_Q(V-Nb)/N)+1)$

$dG = V dp$ @ const $\tau, N \Rightarrow G_g - G_l = \int V dp$

Grand Pot: $\Omega = U - TS - \mu N = -PV = -k_B T \log \mathcal{Z}$

$\langle N\rangle = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V};\ S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu}$

$\mathcal{Z} = \prod_i \mathcal{Z}_i$ for $\mathcal{Z} = \sum_{n_i} \exp(-\beta(\varepsilon_i - \mu_i) n_i)$

Density of States & Dispersion Relns

$\varepsilon = \varepsilon(E)$

$g(\varepsilon) = \int \delta(\varepsilon - \varepsilon(\vec k)) \frac{d^d k}{(2\pi)^d}$

(3D): $g(\varepsilon) = \frac{V}{(2\pi)^3} \int_{\varepsilon(\vec k) = \varepsilon} \frac{dS}{|\nabla_k \varepsilon(\vec k)|} \propto \varepsilon^{1/2}$

Disp. $\varepsilon = \frac{\hbar^2 k^2}{2m} \to g(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$

(2D): $g(\varepsilon) = \frac{Am}{2\pi\hbar^2} \propto 1$

(1D): $g(\varepsilon) = \frac{L}{\pi\hbar}\sqrt{\frac{m}{2\varepsilon}} \propto \varepsilon^{-1/2}$

$k_F = \left(\frac{3\pi^2 n}{g}\right)^{1/3}$ $\quad \int d\Omega = 4\pi$ $\quad$ For degen g: DOS: g · DOS w/o degen

Misc.

Binom Moment trick: $\langle n^k\rangle$. diff wrt. x k times, set x = 1 solve.

$(1+x)^N = \sum_{n=0}^{N} \binom{N}{n} x^n$

Hyperbolic Trig

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh^2 x - \sinh^2 x = 1$

Math

$\log(xy) = \log x + \log y$

$\log(x/y) = \log x - \log y$

$\log x^n = n \log x$

$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$\frac{1}{1-x} \approx 1 - x + x^2 + \dots$ for $|x| < 1$

$\frac{d \ln x}{dx} = \frac{1}{x};\ \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$\frac{d}{dx}(uv) = \frac{du}{dx} v + \frac{dv}{dx} u$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$

$\langle N\rangle = -\left(\frac{\partial \Phi}{\partial \mu}\right)_{T,V}$

$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$

$\ln(N!) \approx N \ln N - N$

$\Gamma(z+1) = z\Gamma(z)$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\cancel{S_n = \frac{a(r^n - 1)}{r - 1}}$

$\Phi = -kT \ln \mathcal{Z}$

$S = \frac{a}{1-r}$ $|r| < 1$

Σ sum of inf. geo ser.

$x = \frac{h\nu}{kT} \to \nu = \frac{kT}{h}x$ $\quad \int \frac{\nu^3}{\exp(h\nu/kT) - 1}$ $\quad \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$

$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

$\int_{-\infty}^{\infty} dx\, x e^{-x^2} = \frac{1}{2}$

$\int_{-\infty}^{\infty} dx\, x^2 e^{-x^2} = \sqrt{\frac{\pi}{4}}$

$\int u\, dv = uv - \int v\, du$

$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$

Adiabatic Magnetocaloric Eff.

$\Delta T_{ad} = -\int_{H_0}^{H_1} \left(\frac{T}{C(T,H)}\right)_H \left(\frac{\partial M(T,H)}{\partial T}\right)_H dH$

Superconductors

· 0 elec. res. below T_c

· Meissner Eff. expels all internal mag. fields.

$dU = TdS + \mu_0 H dM$

Type I: above H_c, superconductivity lost abruptly

Type II: b/w H_{c1}, H_{c2}, mixed state, mag. flux penetrates in quantized vortices

In d dims, # states b/w $\vec p$ & $\vec p + d\vec p$: $\frac{V}{h^d} d^d p$

Ultrarelativistic $E = pc$ DOS $\propto E^2$

Barometric Formula

$P = P_0 \exp\left(\frac{-g_0 M (h - h_0)}{R T_{M,b}}\right)$

$\langle(\Delta N)^2\rangle = \langle N^2\rangle - \langle N\rangle^2$

$Z_1 = \frac{V}{\lambda_{th}^3}$ $\quad \lambda_{th} = \frac{h}{\sqrt{2\pi m k_B T}}$

Particle separation: $l \sim n^{-1/3} = \left(\frac{N}{V}\right)^{-1/3}$

$\lambda_{th} \ll l \to$ classical

$\lambda_{th} \gtrsim l \to$ quantum

Fermions @ T = 0: $P \sim \frac{E_{tot}}{V} \neq 0$

Monatomic Ideal Gas

$\gamma = \frac{C_P}{C_V} = \frac{5}{3}$

Isentropic Eq. of Path: $PV^\gamma = \text{const.}$

$W = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1}$

$P = \frac{2}{3} \cdot \frac{U}{V} = \frac{2}{5} n E_F$

$P_{deg.} \sim \frac{\hbar^2}{m_e} n_e^{5/3}$ $\quad P_{grav.} \sim \frac{GM^2}{R^4}$; $R \propto M^{-1/3}$

Earth Temp from Rad Bal.

$P_{in} = \pi R_E^2 \cdot \sigma T_{sun}^4 \cdot \left(\frac{R_{sun}}{r_{sun-earth}}\right)^2$

$P_{out} = 4\pi R_E^2 \cdot \sigma T_{Earth}^4$

$P_{in} = P_{out} \to T_{Earth} \sim 287\,K$

Mermin-Wagner Thm

(2D) $\int_0^\infty \frac{g(E)\, dE}{\exp(E/k_B T) - 1} \to \infty \Rightarrow T_c = 0$

↳ No BEC in ideal 2D sys of bosons

Photons are bosons, but # not conserved → $\mu = 0$

Surface Tension & Droplets

σ: energy per unit area

$\Delta P = P_{in} - P_{out} = \frac{2\sigma}{R}$

$a = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$

Stefan-Boltzmann Law for Power

$P = \sigma_B A T^4$ Blackbody Rad: $P = \frac{1}{3}\frac{U}{V}$

Photon Gas: $U = aVT^4$

Avg. # photons in cav: $n = \frac{16\pi \zeta(3) k_B^3}{c^3 h^3} T^3$ $\quad \zeta(3) = 1.202$

Fermi Stuff: $\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 N}{gV}\right)^{2/3}$

$N = \frac{gV}{6\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}$

$p_F = \hbar k_F = \hbar\left(\frac{6\pi^2 N}{gV}\right)^{1/3}$