MATH 110 Crib Sheet (MT 1) DIVIT RAWAL

- a linear restriction reduces $\dim V$ by 1 (proof by Rank-Nullity)
- Vector subspace is (closed under) sum of subspaces is the smallest containing subspace
  - → Vector Addition
  - → Scalar Multiplication
  - → Contains $\vec{0}$

Direct Sum: $V_1 \oplus \dots \oplus V_m$ is a direct sum if $\forall \vec{v} \in V_1 \oplus \dots \oplus V_m$ = a unique sum of $\vec{v}_1 + \dots + \vec{v}_m$ for $\vec{v}_k \in V_k$

→ iff $\vec{0} = \vec{v}_1 + \dots \vec{v}_m$, $\vec{v}_k \in V_k \Rightarrow \vec{v}_k = \vec{0}$

→ $U \oplus W \Leftrightarrow U \cap W = \{0\}$

Span $\{\vec{v}_1 \dots \vec{v}_m\}$ is smallest subspace of V containing all $\vec{v}_k$

$\vec{v}_1, \dots \vec{v}_m$ are LI iff $a_1\vec{v}_1 + \dots + a_m\vec{v}_m = \vec{0} \Rightarrow a_i = 0$

For $\vec{v}_1, \dots, \vec{v}_m$ is LD $\in V$, $\exists k \in \{1, 2, \dots, m\}$ s.t. $\vec{v}_k \in \text{span}(\vec{v}_1, \dots \vec{v}_{k-1})$ & span$\{\vec{v}_1, \dots \vec{v}_{k-1}\}$ = span $\{\vec{v}_1, \dots \vec{v}_m\}$

$\boxed{\text{Every LI list in } V \text{ can be extended to a basis of } V}$ For $U \subset V$, $\exists W \subset V$ s.t. $V = U \oplus W$

$U \subset V$ & $\dim U = \dim V \Rightarrow U = V$ $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$

Linear Map $T: V \to W \in \mathcal{L}(V, W)$

→ additivity $T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$

→ homogeneity $\lambda T v = T(\lambda \vec{v})$

For $\vec{v}_1, \dots, \vec{v}_n$ is a basis of V & $\vec{w}_1, \dots, \vec{w}_n$ $\exists$ a unique T s.t. $T\vec{v}_k = \vec{w}_k$. $\mathcal{L}(V, W)$ is a vector space.

Linear Maps are Associative $T(\vec{0}) = \vec{0}$

null $T = \{\vec{v} \in V \mid T\vec{v} = \vec{0}\} \subset V$ Injectivity: $T\vec{u} = T\vec{v} \Rightarrow \vec{u} = \vec{v}$ injectivity $\Leftrightarrow$ null $T = \{\vec{0}\}$

range $T = \{T\vec{v} \mid \vec{v} \in V\} \subset W$ Surjectivity: range $T = W$ dim = 0

FTLM: $\dim V = \dim \text{null} T + \dim \text{range} T$ $\dim V > \dim W \Rightarrow \nexists$ injective $T: V \to W$

A is $m \times n \Rightarrow m$ rows, n cols $\dim V < \dim W \Rightarrow \nexists$ surjective $T: V \to W$

$T\vec{v}_k = A_{1,k}\vec{w}_1 + \dots + A_{m,k}\vec{w}_m$ $AB_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{r,k}$ Ab for $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 A_{\cdot,1} + \dots + b_n A_{\cdot,n}$

= Column rank: dim span cols A

Row rank: dim span rows

$\boxed{\text{Isomorphism } T \text{ is bijective} \Leftrightarrow \text{invertible}}$

→ Inverse is unique $TT^{-1} = \mathbb{I} = T^{-1}T$

→ injectivity $\Leftrightarrow$ surjectivity iff $\dim V = \dim W$

V isomorphic to W iff $\dim V = \dim W$

dim range T = rank T $(AB)^{-1} = B^{-1}A^{-1}$ $(AB)^T = B^T A^T$ $M(T)_{\beta'}^{\beta'} = M(I)_{\beta}^{\beta'} M(T)_{\beta}^{\beta} (M(I)_{\beta}^{\beta'})^{-1}$

Linear Functional: map $V \to F$ V' (dual space of V) = $\mathcal{L}(V, F)$ $\dim V' = \dim V$

$\boxed{\text{Dual Basis: } \varphi_i(\vec{v}_j) = \delta_{ij}}$ for $\vec{v}_j$ is a basis of V $\vec{v} = \varphi_1(\vec{v})v_1 + \dots + \varphi_n(\vec{v})v_n$

Dual Map: $T' \in \mathcal{L}(W', V')$ s.t. $T'(\varphi) = \varphi \circ T$ $\forall \varphi \in W'$ $\dim U^0 = \dim V - \dim U$ for $U \subseteq V$

☆ For $U \subseteq V$, U^0 (the annihilator of U) = $\{\varphi \in V' \mid \varphi(u) = 0 \ \forall u \in U\}$, $U^0 \subseteq V'$

$U^0 = \{0\} \Leftrightarrow U = V$, $U^0 = V' \Leftrightarrow U = \{0\}$ null $T' = (\text{range} T)^0$ T injective $\Leftrightarrow T'$ surjective

T surjective $\Leftrightarrow T'$ injective $M(T') = (M(T))^t$, dim null T' = dim null T + dim W - dim V

dim range T' = dim range T range $T' = (\text{null} T)^0$ $U \subseteq V$ is invariant under T if $Tu \in U$ $\forall u \in U$

eigenvalue λ if $\exists \vec{v} \in V$ s.t. $T\vec{v} = \lambda\vec{v}$ ← eigenvector eigenvectors of T are LI ↑ corr. to diff e-vals

$T^0 = \mathbb{I}$ null T & range T are invariant under T, $p(T)$ same

Ker $T = \{0\}$ Proving Injectivity: ass. $T\vec{v}_1 = T\vec{v}_2$ for $\vec{v}_1, \vec{v}_2 \in V$. Show $T(\vec{v}_1 - \vec{v}_2) = 0$. Show $\vec{v}_1 - \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_1 = \vec{v}_2$

Proving Surjectivity: for $\vec{w} \in W$, show $\exists \vec{v} \in V$ s.t. $T\vec{v} = \vec{w}$. Range $T = W = \text{Im} T$

Row of matrix represents coords of old basis in terms of new basis.

Col of matrix represents coords. of new basis in terms of old basis

Operator $T: V \to V$ $U \subset V$ invariant if $Tu \in U$ $\forall u \in U$ λ is eig-val of T if $\exists v \neq 0 \in V$ s.t. $T\vec{v} = \lambda\vec{v}$

λ an e-val of $T \Leftrightarrow T - \lambda I$ not invertible $\Leftrightarrow T - \lambda I$ not surj $\Leftrightarrow T - \lambda I$ not inj at most $\dim V$ e-vals

$\exists!$ monic polynomial s.t. $p(T) = 0$ & $\deg p \leq \dim V$ Minimal Polynomial: ! monic polynomial of smallest deg. s.t. $p(T) = 0$

T not invertible $\Leftrightarrow$ const term of min poly = 0

Upper Triangular: $\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ $\Leftrightarrow M(T)$ w.r.t. $v_1, \dots, v_n$ is upper triangular $(x - \lambda_1) \dots (x - \lambda_m)$

$\Leftrightarrow$ span$(v_1, \dots, v_n)$ invariant under V

$\Leftrightarrow$ span$(v_1, \dots, v_k) \ni Tv_k$

$\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = ad - bc$

$T \in \mathcal{L}(V)$ & T upper tri w/ diagonal $\lambda_1, \dots, \lambda_n \Rightarrow (T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$

T has upper tri matrix iff min poly $(T) = (z-\lambda_1)\cdots(z-\lambda_m)$ for $\lambda_i \in F$

$E(\lambda, T) = \text{null}(T - \lambda I) = \{v \in V \mid Tv = \lambda v\}$. $E(\lambda_1, T) + \dots + E(\lambda_m, T)$ is dir. sum

$\dim(E(\lambda_1,T)) + \dots \dim(E(\lambda_m,T)) \le \dim V$

T is diagonalizable $\Leftrightarrow$ V has a basis of e-v of T $\Leftrightarrow V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T) \Leftrightarrow \dim V = \sum_i \dim(E(\lambda_i, T))$

If $[S,T] = 0$, $E(\lambda, S)$ is invariant under T. $[S,T] = 0 \Leftrightarrow$ simoultaneously diag.

Inner Product Axioms
- Positivity: $\langle v, v \rangle \ge 0 \ \forall v$
- Definiteness: $\langle v, v \rangle = 0$ iff $v = 0$
- Additivity in 1st slot: $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- Homogeneity in 1st slot: $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$
- Conj Sym: $\langle u, v \rangle = \overline{\langle v, u \rangle}$

Inner Product Props
- For fixed v, $u \to \langle u, v \rangle$ is a lin map.
- $\langle 0, v \rangle = 0$, $-\langle v, 0 \rangle = 0$
- $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$
- $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$

If $\langle u, v \rangle = 0 \Rightarrow \|u+v\|^2 = \|u\|^2 + \|v\|^2$

$\|v\| = \sqrt{\langle v, v \rangle}$

Norm Props
- $\|v\| = 0$ iff $v = 0$
- $\|\lambda v\| = |\lambda| \|v\|$
- $\langle u, v \rangle = 0 \Rightarrow$ orthogonal
- $\langle 0, 0 \rangle = 0$ & $\langle 0, v \rangle = \langle v, 0 \rangle = 0 \ \forall v \in V$

Ortho Decomp set $v, u \in V$ w/ $v \neq 0$

$c = \frac{\langle u, v \rangle}{\|v\|^2}$ & $w = u - \frac{\langle u, v \rangle}{\|v\|^2} v$ Then $u = cv + w$ & $\langle w, v \rangle = 0$

Cauchy-Schwarz Ineq

$|\langle u, v \rangle| \le \|u\| \|v\|$

Triangle Ineq.

$\|u+v\| \le \|u\| + \|v\|$

$\|$-gram Eq

$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

$e_1, \dots, e_n$ orthonormal $\Rightarrow \|a_1 e_1 + \dots + a_m e_m\|^2 = |a_1|^2 + \dots + |a_m|^2$

Bessel's Ineq

$v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n$

$|\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_m \rangle|^2 \le \|v\|^2$

For φ a lin. functional on V, $\exists!\, v \in V$ s.t. $\varphi(u) = \langle u, v \rangle$

$U \subset V \Rightarrow U^\perp = \{v \in V \mid \langle u, v \rangle = 0 \ \forall u \in U\} \subset V$ $\quad \{0\}^\perp = V$ & $V^\perp = \{0\}$

$U \subset V \Rightarrow U \cap U^\perp = \{0\}$ $\quad V = U \oplus U^\perp$ $\quad U = (U^\perp)^\perp$

For $T \in \mathcal{L}(V, W)$ T^* s.t. $\langle Tv, w \rangle = \langle v, T^* w \rangle$ $\quad (S+T)^* = S^* + T^*$ $\quad (\lambda T)^* = \bar{\lambda} T^*$ $\quad (T^*)^* = T$, $(ST)^* = T^* S^*$ $\quad I^* = I$ $\quad T$ inv $\Rightarrow$ T^* inv & $(T^*)^{-1} = (T^{-1})^*$

null $T^* = (\text{range } T)^\perp$, range $T^* = (\text{null } T)^\perp$ $\quad T^* =$ conj. Transpose of T $\quad T = T^* \Rightarrow$ self adj $\Rightarrow$ real e-vals

null $T = (\text{range } T^*)^\perp$, range $T = (\text{null } T^*)^\perp$ $\quad \langle Tv, v \rangle = 0 \ \forall v \Rightarrow T = 0$, $\langle Tv, v \rangle \in \mathbb{R} \Rightarrow T = T^*$ $\quad$ normal if $TT^* = T^*T$

T normal $\Rightarrow$ e-vecs of T corr. diff e-vals are orthogonal. $T = T^* \Leftrightarrow T$ diagonalizable $\Leftrightarrow V$ has an o-n basis of e-vecs of T ($TT^* = T^*T$ (in $\mathbb{C}$))

Generalized E-vecs & Nilpotent Ops

null $T^{\dim V}$ = null $T^{\dim V + 1} = \dots$

$\{0\} = \text{null } T^0 \subseteq \text{null } T \subseteq \text{null } T^k \subseteq \dots$ $\quad$ null T^m = null $T^{m+1} \Rightarrow$ null T^{m+1} = null $T^{m+2} \dots$ $\quad V = \text{null } T^{\dim V} \oplus \text{range } T^{\dim V}$

v is a gen e-vec if $(T - \lambda I)^k v = 0$ for $k \in \mathbb{Z}_{>0}$ $\quad$ gen e-vec corr. to unique e-val & are lin indep.

$T^k = 0 \Rightarrow$ nilpotent. $T \in \mathcal{L}(V)$ nilpotent $\Rightarrow T^{\dim V} = 0$; T nilpotent $\Rightarrow 0$ is only e-val

T nilpotent $\Leftrightarrow$ min poly $= z^m$ for $m \in \mathbb{Z}_{>0} \Leftrightarrow \exists$ basis of V s.t. $T = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$ $\Leftarrow$ (in $\mathbb{C}$) $\quad G(\lambda_k, T)$ inv. under T

$G(\lambda, T) = \{v \in V \mid (T - \lambda I)^k v = 0$ for some $k \in \mathbb{Z}_{>0}\} = \text{null}(T - \lambda I)^{\dim V}$ $\quad V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$

multiplicity of λ = dim null $(T - \lambda I)^{\dim V}$ $\quad \sum$ mult $\lambda_i = \dim V$ $\quad$ geo. mult = $\dim E(\lambda, T)$ $\quad$ alg. mult = $\dim(G(\lambda, T))$

char poly. = min poly w/ each e-val term ^ mult of λ $\quad$ mult of λ = # times on diagonal

Jordan Stuff

basis is Jordan basis if $M(T, b) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$ for $A_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_k \end{pmatrix}$

Process

(1) E-vals of M (2) alg & geo mults (3) dim of $G(\lambda, M)$

If λ has 1 e-vec & alg mult > 1, solve for gen. e-vecs.

Basis (concat for Q) is Jord chain of e-vecs in order

alg mult − geo mult = # gen e-vecs

Riesz Rep. Thm

$\exists!\, v$ s.t. $\varphi(u) = \langle u, v \rangle$ $\quad \langle u, v \rangle \in \mathbb{C} = \sum_{i=1} u_i \bar{v}_i$

Gen e-vecs: $(A - \lambda I)^k = v \Rightarrow v_{i+1} = (A - \lambda I) v_i$

Honest if: $(A - \lambda I)^k v = 0$ $\quad$ char poly $(T) = 0$

Gram-Schmidt

$e_1 = v_1$

$e_2 = v_2 - e_1 \frac{\langle v_2, e_1 \rangle}{\langle e_1, e_1 \rangle}$

⋮

Every nilpotent op. has a Jordan basis

MultiLin & Determinants

bilinear form: $\beta: V \times V \to F$ s.t. $v \mapsto \beta(v, u)$ & $v \mapsto \beta(u, v)$ are lin funcs. $\forall u$ $\quad V^{(2)}$

$M(\beta)_{j,k} = \beta(e_j, e_k)$

sym if $\rho(u, w) = \rho(w, u) \ \forall u, w$

$A = A^T \Leftrightarrow$ symm $\quad$ alt if $\alpha(v, v) = 0$ & $\alpha(u, w) = -\alpha(w, u)$

$V^{(2)} = V^{(2)}_{alt} \oplus V^{(2)}_{sym}$

$\beta \in V^{(m)}$ is m-linear if lin in every slot

$V^{(m)}_{alt}$ if $\alpha(v_1, \dots, v_m)$ if $v_i = v_j$

If $m \ge \dim V$, 0 is the only alt m-lin form on V

swap order of any 2 vecs = factor of -1

sign of perm = $(-1)^N$ for N = # transpositions

$\dim V_{alt}^{(\dim V)} = 1$

$\det A = \sum_{\sigma \in \text{perm } n} (\text{sign}(\sigma)) A_{\sigma_1, 1} \cdots A_{\sigma_n, n}$

det of upper tri. matrix = $\lambda_1 \cdots \lambda_n$

invertible $\Leftrightarrow \det \neq 0$

$\det(T^{-1}) = \frac{1}{\det T}$

$\det A^T = \det A$

$\det T^* = \overline{\det T}$

$\det(ST) = \det(T)\det(S)$

λ is e-val of T iff $\det(T - \lambda I) = 0$

(Co)fac Mat

root is of form $\frac{\text{fac of last term}}{\text{fac of first term}}$