PHYSICS 105 - DIVIT RAWAL

$m\ddot{x} = F(t, x, v)$

Eqns of Motion

- If $F = F(t)$

$m\frac{dv}{dt} = F(t)$

$v(t) = v_0 + \frac{1}{m}\int^t F(t')\,dt'$

$x(t) = x_0 + \int_{t_0}^{t} v(t')\,dt'$

- If $F = F(v)$

$m\frac{dv}{dt} = F(v) \Rightarrow \frac{dv}{F(v)} = \frac{dt}{m}$

$\rightarrow \int_0^{v(t)} \frac{dv'}{F(v')} = \frac{1}{m}(t - t_0)$

$x(t) = \int_0 v(t)\,dt$

- If $F = F(x)$

$m\frac{dv}{dt} = F(x)$ $\quad \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$

$\rightarrow mv\frac{dv}{dx} = F(x)$

$m\int v\,dv = \int F(x)\,dx \Rightarrow \frac{1}{2}mv^2 = \int F(x)\,dx$

Kinetic Energy = Work Done

$F \doteq -\frac{dV}{dx} \rightarrow \frac{1}{2}mv^2 + V(x) = c$

Conservation of Energy

$\int F dt = \Delta p$

Equilibrium : $F(x_0) = 0$

Let $x = x_0 + \delta \rightarrow F(x) = F(x_0) + (x - x_0)F'(x_0) + \dots$

$\approx (x - x_0)F'(x_0)$

If $k \doteq -F'(x_0)$,

$F(x) \simeq -k(x - x_0)$ ← harmonic oscillator!

$\Rightarrow m\ddot{x} = -k(x - x_0)$

$\rightarrow \ddot{x} + \omega^2(x - x_0) = 0$ for $\omega^2 \doteq \frac{k}{m}$

Stable if $F'(x_0) < 0$ $\quad F \simeq -k(x - x_0)$

Unstable if $F'(x_0) < 0$ $\quad F \simeq k(x - x_0)$

$F = -\frac{dV}{dx} \Rightarrow$ eq @ $\frac{dV}{dx} = 0$

$V(x) \simeq V(x_0) + \frac{1}{2}V''(x_0)(x - x_0)$

$\rightarrow F \simeq -V''(x_0)(x - x_0)$

$\rightarrow k = -V''(x_0)$

$V'' > 0 \Rightarrow$ min $\Rightarrow$ stable

$V'' < 0 \Rightarrow$ max $\Rightarrow$ unstable

General Osc. Soln.

$x(t) = A\cos(\omega_0(t - t_0)) + B\sin(\omega_0(t - t_0))$

Normal Modes

$\vec{x} \doteq \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \forall i,\ m_i\ddot{x}_i = F_i(x_1, \dots, x_N)$

Hessian

$K_{ij} \doteq -\frac{\partial F_i}{\partial x_j}\Big|_0 \Rightarrow F_i = -\sum_j K_{ij}x_j$

Then, $M\ddot{\vec{x}} = -K\vec{x}$ for $M = \mathrm{diag}(\{m_i\})$

$\equiv M\ddot{\vec{x}} + K\vec{x} = 0$

Near eq. $x = 0$, $F_i(\vec{x}) = \sum_j \frac{\partial F_i}{\partial x_j}\Big|_0 x_j$

Set. $\vec{x} = \vec{v}\exp(i\omega t)$

$\Rightarrow \ddot{\vec{x}} = -\omega^2\vec{v}\exp(i\omega t)$

$\rightarrow M(-\omega^2\vec{v}\exp(i\omega t)) + K\vec{v}\exp(i\omega t) = 0$

$\Rightarrow (-\omega^2 M + K)\vec{v} = 0 \equiv K\vec{v} = \omega^2 M\vec{v}$

↑ eig. val prob.

Nontrivial if $\det(K - \omega^2 M) = 0$

determining allowed ω^2

If $\vec{x}(t) = \vec{v}\cos(\omega t)$

$\vec{x} = \sum q_n(t)\,v_n$ where

each q_n satisfies

$\ddot{q}_n + \omega_n^2 q_n = 0$

Forced Oscillations

$\ddot{x} + \omega_0^2 x = \frac{f(t)}{m}$

General Structure:

$x(t) = x_{hom}(t) + x_{part}(t)$

$x_{hom}(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

Let $f(t) = F_0\cos(\omega t)$.

$x(t) = A\cos(\omega t)$

w/ $A = \frac{F_0/m}{\omega_0^2 - \omega^2}$

If $\omega \to \omega_0$, $A \to \infty$, energy builds up, large osc.

$\omega \ll \omega_0$, system follows force slowly

$\omega \gg \omega_0$, system can't keep up.

Arbitrary $F(t)$:

Fourier: $f(t) = \sum f_n\cos(n\omega t)$

↳ solve each & add

Green's Fn: $f(t) = \int f(t')\,\delta(t - t')\,dt'$

↳ works b/c DE is linear

$\ddot{G} + \omega_0 G = \delta(t - t')$

$\rightarrow G(t, t') = \frac{1}{\omega_0}\sin(\omega_0(t - t'))\,\Theta(t - t')$

$\rightarrow x(t) = \frac{1}{m\omega_0}\int_{-\infty}^{t}\sin(\omega_0(t - t'))\,f(t')\,dt'$

↑ undamped case, $\quad \Theta(x) \doteq \begin{cases} 0 & \text{else} \\ 1 & \text{if } x > 0 \end{cases}$

Gravitation

for a point mass $\vec{F} = -\frac{GMm}{r^2}\hat{r}$

$d\vec{F} = -\frac{G\,m\,dm}{r^2}\hat{r}$, $\vec{F} = \int d\vec{F}$

Thin Spherical Shell:

$F(r) = \begin{cases} -\frac{GMm}{r^2} & r > R \\ 0 & r \le R \end{cases}$

$V(r) = -\int \frac{G\,m\,dm}{L}$ $\quad F_r = -\frac{dV}{dr}$

center of shell @ origin

$L = \sqrt{R^2 + r^2 - 2Rr\cos(\theta)}$

$V(r) = -\int \frac{G\,m\,dm}{L}$ $\quad dA = 2\pi R^2\sin\theta\,d\theta$

$dm = \sigma\,dA$

set $u = L^2$; $V \propto [(R + r) - |R - r|]$

Solid sphere, uniform density:

$F(r) = \begin{cases} -\frac{GMm}{r^2} & r > R \\ -\frac{GMm}{R^3} r & r < R \end{cases}$

$M(r) = M\frac{r^3}{R^3}$ $\quad$ inside: $F \propto -r$

SHM w/ $\omega^2 = \frac{GM}{R^3}$

Infinite sheet: $\quad F = -2\pi G\sigma m$

symmetry → only vertical force

$L = \sqrt{r^2 + x^2}$ $\quad dF = -\frac{G\,m\,dm}{L^2}\frac{x}{L}$

$\int_0^\infty \frac{r\,dr}{(r^2 + x^2)^{3/2}} = \frac{1}{x}$

sheet w/ circular hole: (rad R)

$F(x) = -\frac{2\pi G\sigma m x}{\sqrt{R^2 + x^2}}$

$F_{\text{sheet above}}$ $\quad F_{disk} = -2\pi G\sigma m\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$

$F_{hole} = F_{sheet} - F_{disk}$

Gauss' Law

$\nabla \cdot g = -4\pi G\rho$

$\oint g \cdot dA = -4\pi G M_{enc}$

$g = -\nabla\Phi$

Vector Calc Stuff

$\nabla \cdot F$ - div $\quad \nabla \times F$ curl

$\nabla^2 = \nabla \cdot \nabla F$, $\int_V (\nabla \cdot F)\,dV = \oint_S F \cdot dA$

$\int_S (\nabla \times F) \cdot dA = \oint_{\partial S} F \cdot dl$

$\nabla \times \nabla f = 0$ $\quad \nabla \cdot (\nabla \times F) = 0$

$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

Plane $ax + by + cz + d = 0$

line $r_0 + rt$

Forced Harmonic Motion

$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$

$x(t) = x_{tr}(t) + x_{steady}(t)$

$x(t) = X \sin(\omega t - \phi)$

$X(\omega) = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$

$\tan\phi = c\omega/(k - m\omega^2)$

$\omega \to 0 \quad X = \dfrac{F_0}{k}$

$\omega \to \infty \quad X = \dfrac{F_0}{m\omega^2}$

Complex Method:

$x(t) = \tilde{A} e^{-i\omega t}$

$\tilde{A} = \dfrac{F_0}{k - m\omega^2 + ic\omega}$

$|\tilde{A}| = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$

Fourier Decomp:

for periodic $F(t)$ w/ period T, $\omega_1 \doteq \dfrac{2\pi}{T}$

$F(t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t)]$

$a_n \doteq \dfrac{2}{T}\int_0^T F(t)\cos(n\omega_1 t)\,dt$

$b_n \doteq \dfrac{2}{T}\int_0^T F(t)\sin(n\omega_1 t)\,dt$

$a_0 \doteq \dfrac{2}{T}\int_0^T F(t)\,dt$

or $a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t)$

$= A_n \sin(n\omega_1 t + \phi_n)$ for

$A_n = \sqrt{a_n + b_n}$

If not periodic

$F(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega) e^{i\omega t}\,d\omega$

w/ $\tilde{F}(\omega) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(t) e^{i\omega t}\,dt$

Gravitation of a Rod

rod on x axis; λ mass density;

test mass m off rod

$dF = \dfrac{G m\, dm}{r^2}$ $\quad dg = G\dfrac{dm}{r^2}$ toward source

$dm = \lambda\, dx$

Variations

$L = T - V + \sum_\alpha \lambda_\alpha f_\alpha$

$S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\,dt$

$\delta S = 0$ for physical path

$q(t) \to q(t) + \epsilon \eta(t) \quad \delta q = \eta$

$\delta\dot{q} = \dfrac{d}{dt}\delta q \quad \delta q(t_1) = \delta q(t_2) = 0$

$\delta S = \int_{t_1}^{t_2}\left(\dfrac{\partial L}{\partial q}\delta q + \dfrac{\partial L}{\partial \dot{q}}\delta\dot{q}\right)dt$

Euler-Lagrange:

holonomic constraint

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = \sum_\alpha \lambda_\alpha \dfrac{\partial f_\alpha}{\partial q_i}$

boundary term $\left[\dfrac{\partial L}{\partial \dot{q}}\delta q\right]_{t_1}^{t_2}$

fixed endpts $= 0$,

free gives conditions

Hamiltonian Mechanics

$p_i = \dfrac{\partial L}{\partial \dot{q}_i} \quad H = T + V$

$H = \sum_i p_i \dot{q}_i - L$

$\dot{q}_i = \dfrac{\partial H}{\partial p_i} \quad \dot{p}_i = -\dfrac{\partial H}{\partial q_i}$

$\dfrac{dH}{dt} = \dfrac{\partial H}{\partial t} = 0 \Rightarrow H$ conserved

$\dfrac{\partial H}{\partial q_i} = 0 \Rightarrow q_i$ cyclic $\Rightarrow p_i$ conserved

$H(q_i, p_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$

$\{f, g\} = \sum_i\left(\dfrac{\partial f}{\partial q_i}\dfrac{\partial g}{\partial p_i} - \dfrac{\partial f}{\partial p_i}\dfrac{\partial g}{\partial q_i}\right)$

$\dfrac{df}{dt} = \{f, H\} + \dfrac{\partial f}{\partial t}$ $\quad f$ conserved $\Downarrow$ $\{f, H\} = 0$

$\{q_i, q_j\} = 0 \quad \{q_i, p_j\} = \delta_{ij}$

$\{p_i, p_j\} = 0$

stationary radius: $\dfrac{\partial V_{eff}}{\partial r}\Big|_{r=r_0} = 0$

Central Force Motion

$F = F(r)\hat{r} \quad V = V(r) \quad \tau = r \times F = 0$

$L =$ const. $\quad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$p_\theta = mr^2\dot{\theta} = \ell \quad \dot{\theta} = \dfrac{\ell}{mr^2}$

$\dfrac{dA}{dt} = \dfrac{1}{2}r^2\dot{\theta} = \dfrac{\ell}{2m}$

$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r)$

$= \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$

for $V_{eff}(r) = V(r) + \dfrac{\ell^2}{2mr^2}$

allowed motion: $E \geq V_{eff}(r)$

turning pts: $V_{eff}(r) = E$

$m\ddot{r} = \dfrac{\ell^2}{mr^3} - \dfrac{dV}{dr} \quad m\ddot{r} = -\dfrac{dV_{eff}}{dr}$

stable if $V''_{eff}(r_0) > 0$

unstable if $V''_{eff}(r_0) < 0$

$\dfrac{dV_{eff}}{dr}\Big|_{r_0} = 0 \equiv \dfrac{dV}{dr}\Big|_{r_0} = \dfrac{\ell^2}{mr_0^3}$

let $u = \dfrac{1}{r}$: $\dfrac{d^2u}{d\theta^2} + u = -\dfrac{m}{\ell^2 u^2}F\left(\dfrac{1}{u}\right)$ for $F(r) = -\dfrac{dV}{dr}$

for $F(r) = -\dfrac{k}{r^2}$, $V(r) = -\dfrac{k}{r}$

$r(\theta) = \dfrac{p}{1 + e\cos\theta} \quad p = \dfrac{\ell^2}{mk} \quad e = \sqrt{1 + \dfrac{2E\ell^2}{mk^2}}$

$H = \dfrac{p_r^2}{2m} + \dfrac{p_\theta^2}{2mr^2} + V(r) \quad p_r = m\dot{r}$

$p_\theta = \ell =$ const.

Near eq. $r = r_0 + \delta r$,

EOM: $m\ddot{\delta r} + U''_{eff}\,\delta r = 0$

Systems of Particles

$M = \sum_i m_i$

$R = \dfrac{1}{M}\sum_i m_i r_i \quad P = M\dot{R} \quad \dfrac{dP}{dt} = F_{ext}^{tot}$

$M\ddot{R} = F_{ext}^{tot} \quad F_{ext}^{tot} = \sum_i F_i^{ext} \quad \rho_i = r_i - R$

$\sum_i m_i \rho_i = 0 \quad \sum_i m_i \dot{\rho}_i = 0 \quad T = T_{cm} + T_{rel}$

$= \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\sum_i m\dot{\rho}_i^2$

$L = \sum_i r_i \times m_i \dot{r}_i$

$= MR \times \dot{R} + \sum_i \rho_i \times m_i \dot{\rho}_i = L_{cm} + L_{rel}$

$\dfrac{dL}{dt} = \tau_{ext}^{tot} \quad V = \sum_i V_i^{ext}(r_i) + \sum_{i<j} V_{ij}(|r_i - r_j|)$

$= \dfrac{1}{2}\sum_i\sum_{j \neq i} V_{ij} \quad E = T + V$

for 2-body:

$\mu = \dfrac{m_1 m_2}{m_1 + m_2} \quad \mu\ddot{r} = -\nabla V(r)$

transl sym: P const

rot sym: L const

time transl sym: E const

Noninertial Frames

frame origin has accel A

$F_{pseudo} = -mA \quad ma' = F - mA$

for rot w/ ang. vel Ω

for any vector Q $\left(\dfrac{dQ}{dt}\right)_{inertial} = \left(\dfrac{dQ}{dt}\right)_{rotating} + \Omega \times Q$

$v = v' + \Omega \times r \quad a = a' + 2\Omega \times v' + \Omega \times (\Omega \times r) + \dot{\Omega} \times r$

if origin transl. w/ accel A: $+ A$ to above

rot. frame eq. of motion

$ma' = F - mA - 2m\Omega \times v' - m\Omega \times (\Omega \times r) - m\dot{\Omega} \times r$

Psendo-forces: $F_{tr} = -mA \quad F_{cor} = -2m\Omega \times v'$

$F_{cent} = -m\Omega \times (\Omega \times r)$

$= m\Omega^2 r$ if $r \perp \Omega$ $\quad$ w/ dist ρ from rot axis, $F_{cent} = m\Omega^2\rho$

$F_{Euler} = -m\dot{\Omega} \times r$

$V_{eff} = V - \frac{1}{2}m|\Omega \times r|^2 = V - \frac{1}{2}m\Omega^2\rho^2$

Equil @ $\dfrac{dV_{eff}}{dq} = 0$ $\quad$ stable if $V'_{eff} > 0$

$L = \frac{1}{2}mv'^2 + mv' \cdot (\Omega \times r) + \frac{1}{2}m|\Omega \times r|^2 - V$

Rigid Bodies

$|r_i - r_j| =$ const

$I = \sum_i m r_{i\perp}^2 \quad v_i = V_{cm} + \omega \times \rho_i$

$I = \int r_\perp^2\,dm \quad I = I_{cm} + Md^2$

for planar bodies

$I_z = I_x + I_y$ ← flat obj in x,y plane

$I_{ij} = \int (r^2\delta_{ij} - x_i x_j)\,dm \quad L = I\omega \quad \tau = I\alpha$

Parallel axis thm. If I_{cm} is mom of inertia, moment about a parallel axis displaced by d is $I_{cm} + Md^2$. rolling w/o slip: $a = R\alpha$

Coupled Oscillators

$M\ddot{q} + Kq = 0$ — M is mass mat. K is stiffness mat

Ansatz: $q(t) = a\exp(i\omega t)$

$(K - \omega^2 M)a = 0$ — $\det(K - \omega^2 M) = 0$ ← freq. cond.

$(K - \omega^2 M)a = 0$ ← mode shapes, solve $\forall \omega$

$q(t) = \sum_r C_r a_r \cos(\omega_r t + \phi_r)$

$L = \frac{1}{2}\dot{q}^T M \dot{q} - \frac{1}{2} q^T K q$

E-L gives $M\ddot{q} + Kq = 0$

Near eq $K_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{eq}$

$V \approx V_0 + \frac{1}{2}\sum_{ij} K_{ij}\eta_i\eta_j \quad \eta_i = q_i - q_{eq}$

$a_r^T M a_s = 0$ for $r \neq s$

normalized $\equiv a_r^T M a_s = \delta_{rs}$

choose coords along normal modes s.t. $\ddot{Q}_r + \omega_r^2 Q_r = 0$

Beats: if 2 close freqs are excited

$\omega_{beat} = |\omega_2 - \omega_1| \quad T_{beat} = \frac{2\pi}{|\omega_2 - \omega_1|}$

$T_{transfer} = \frac{T_{beat}}{2}$

$\omega^2 > 0$ stable

$\omega^2 = 0$ free mode

$\omega^2 < 0$ unstable

Some Diffeq. Ansatzes

$\frac{dy}{dx} = f(x)g(y) \to \frac{dy}{g(y)} = dx f(x)$

$y' + p(x)y = q(x) \to (\mu y)' = \mu q$ for $\mu = \exp(\int p(x)dx)$

$y' = F(\frac{y}{x}) \to y = vx$

$ay'' + by' + cy = 0 \to y = e^{rx}$ for $ar^2 + br + c = 0$

Special Relativity

$\beta = \frac{v}{c}, \gamma = \frac{1}{\sqrt{1-\beta^2}}$

S' moves sp. v in $+x$ dir rel to S.

$S \to S'$: $ct' = \gamma(ct - \beta x)$

$x' = \gamma(x - \beta ct)$

$y' = y \quad z' = z$

Poisson Brackets

$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$ ← $= 0$ if no explicit time dep.

$-\{g, f\} = \{f, g\}$

if $\{f, H\} = 0$, f is conserved.

$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$

More post-MT stuff

Rolling w/o slipping: $\Delta x = R\Delta\theta$

$T = \frac{1}{2} I_{contact}\omega^2 = \frac{1}{2}(I_{CM} + Md^2_{contact,CM})\omega^2$

If central force is inv-sq in distance,

$\frac{A}{r} = 1 + \varepsilon\cos\theta$ where $A = \frac{\ell^2}{mk}$ and

$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$ is eccentricity.

ellipse when $E < 0$ $(\varepsilon < 1)$

parabola when $E = 0$ $(\varepsilon = 1)$

hyperbola when $E > 0$ $(\varepsilon > 1)$

$\alpha = 2\sin^{-1}\left(\frac{-1}{\varepsilon}\right) = 2\tan^{-1}\left(\frac{k}{mv_0^2 b}\right)$ if $\ell = mv_0 b$

Coupled Oscillations

$L = T - V \quad T = \frac{1}{2}\dot{q}^T M \dot{q} \quad V = \frac{1}{2} q^T K q$

Elliptic Orbits

$\vec{F} = -\frac{k}{r^2}\hat{r}$

$r(\theta) = \frac{p}{1 + \varepsilon\cos\theta} \quad p = \frac{\ell^2}{mk}$

$a = -\frac{k}{2E}$ (semi-major axis ~~length~~)

$p = a(1 - \varepsilon^2)$; $\ell^2 = mka(1 - \varepsilon^2)$

periapsis: $r_{min} = a(1 - \varepsilon) = \frac{p}{1 + \varepsilon}$

apoapsis: $r_{max} = a(1 + \varepsilon) = \frac{p}{1 - \varepsilon}$

Period: $k = GMm$ (grav. orbit)

$T^2 = \frac{4\pi^2}{GM}a^3 \Rightarrow T = 2\pi\sqrt{\frac{ma^3}{k}}$

$v^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$

$b = \frac{p}{\sqrt{1-\varepsilon^2}} = \frac{\ell}{\sqrt{2m|E|}}$

Ellipse $I_z = \frac{1}{4}M(a^2 + b^2)$, $I_y = \frac{1}{4}Ma^2$, $I_x = \frac{1}{4}Mb^2$

Calc Stuff

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan(\frac{x}{a})$

$\int u\,dv = uv - \int v\,du$

$\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \frac{\theta^2}{2}$

$\tan\theta \approx \theta \quad e^{ix} = \cos x + i\sin x$

$(1+x)^n = 1 + nx$

$e^x \approx 1 + x$

$\sin(2x) = 2\sin x\cos x$

$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\sin(a \pm b) = \sin a\cos b \pm \cos a\sin b$

$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$

$A\sin x + B\sin x = R\sin(x + \phi)$

$R = \sqrt{A^2 + B^2} \quad \tan\phi = \frac{B}{A}$

Coords. $x = r\cos\theta, y = r\sin\theta$

$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}(\frac{y}{x})$

$dV = r\,dr\,d\theta\,dz$

$x = r\sin\phi\cos\theta \quad y = r\sin\phi\sin\theta$

$z = r\cos\phi \quad r = \sqrt{x^2 + y^2 + z^2}$

$\theta = \tan^{-1}(\frac{y}{x}) \quad \phi = \cos^{-1}(\frac{z}{r})$

$dV = r^2\sin\phi\,dr\,d\theta\,d\phi$

$\vec{a}\cdot\vec{b} = |a||b|\cos\theta \quad |\vec{a}\times\vec{b}| = |a||b|\sin\theta$

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \quad \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})$

$\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \dot{\vec{a}}\cdot\vec{b} + \vec{a}\cdot\dot{\vec{b}}$

$\frac{d}{dt}(\vec{a}\times\vec{b}) = \dot{\vec{a}}\times\vec{b} + \vec{a}\times\dot{\vec{b}}$

Known Integrals

$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x)$

$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})$

$\int \ln x\,dx = x\ln x - x$

$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x)$

$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x\cos x)$

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin(\frac{x}{a})$

Known Moments of Inertia

point mass dist r from axis $I = mr^2$

thin rod, center, perp to rod $I = \frac{1}{12}ML^2$

thin rod, end, perp to rod $I = \frac{1}{3}ML^2$

solid disk/cyl, central axis $I = \frac{1}{2}MR^2$

hoop, central axis $I = MR^2$

solid sphere, through center $I = \frac{2}{5}MR^2$

thin spherical shell, center $I = \frac{2}{3}MR^2$

Rigid Body Stuff

for rigid body in body-fixed principal axis frame

$L = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$.

Euler Eqns.

$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = N_1$

$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = N_2$

$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = N_3$

for torque-free rigid body

$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = \text{const}$

$L^2 = \text{const} = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$

rotation about largest or smallest principal axis is stable, about intermediate is unstable

Effective gravity Earth

rot @ Ω.

$\vec{g}_{eff} = \vec{g} - \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$

Coriolis $\vec{a}_{cor} = -2\vec{\Omega}\times\vec{v}$

for horizontal motion at latitude λ, vertical comp $\Omega_z = \Omega\sin\lambda$

$a_{cor} \approx 2\Omega v\sin\lambda$

Examples

Block & Wedge

no friction wedge↔block, block ↔ floor

Find a_M, N, a_m rel to M.

N is norm force b/w m & M, $M\ddot{x}_M = \frac{N\sin\theta}{M}$. In frame moving w/ M, forces on m are mg, N, & inertial ma_M

$N + ma_M\sin\theta - mg\cos\theta = 0$

$mg\sin\theta + ma_M\cos\theta = Ma_M$

gives $N = \frac{mMg\cos\theta}{M + m\sin^2\theta}$ →

$a_M = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$

$\to mg\sin\theta + \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$

$\to a_m = g\sin\theta + \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$

$= \frac{(m+M)g\sin\theta}{M + m\sin^2\theta}$

Coriolis Force on Projectile

projectile fired horizontally east @ latitude λ. Find direction & mag of Coriolis acceleration. choose $\hat{x}$ east, $\hat{y}$ north, $\hat{z}$ up

$\vec{a}_c = -2\vec{\Omega}\times\vec{V}$

at latitude λ, Earth velocity vector is

$\vec{\Omega} = \Omega\cos\lambda\hat{y} + \Omega\sin\lambda\hat{z}$

$\vec{V} = v\hat{x}$. $\vec{a}_c = -2(\Omega\cos\lambda\hat{y} + \Omega\sin\lambda\hat{z})\times v\hat{x}$

$\hat{y}\times\hat{x} = -\hat{z}$, $\hat{z}\times\hat{x} = \hat{y}$

$\to \vec{a}_c = 2\Omega v\cos\lambda\hat{z} - 2\Omega\sin\lambda\hat{y}$

Center of Mass

two particles of mass m_1, m_2, pos given by r_i. $r = r_1 - r_2$

Angular momentum wrt CoM is?

$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$ so $r_{1c} = r_1 - R$, $r_{2c} = r_2 - R$

$r_{1c} = -\frac{m_2 r}{m_1 + m_2}$

$r_{2c} = \frac{m_1 r}{m_1 + m_2}$

$\vec{L} = r_{1c}\times m_1\dot{\vec{r}}_{1c} + r_{2c}\times m_2\dot{\vec{r}}_{2c}$

$= m_1\left(\frac{m_2}{m_1+m_2}\right)^2\vec{r}\times\dot{\vec{r}}$

$+ m_2\left(\frac{m_1}{m_1+m_2}\right)^2 r\times\dot{\vec{r}}$

$\mu(\vec{r}\times\dot{\vec{r}})$

Work-Energy

Show W-E Thm is always rel when face strikes plane. correct in CM frame, i.e. if $\vec{r}_{ic}, \vec{v}_{ic}$ are the pos & vel of m_i in CM frame,

$$\sum_i\int\vec{F}_i\cdot d\vec{r}_{ic} = \Delta\sum_i\frac{m_i v_{ic}^2}{2}.$$

In the inertial frame $\sum_i\int\vec{F}_i\cdot d\vec{r}_i = \Delta\sum_i\frac{m_i v_i^2}{2}$.

using $\vec{r}_i = \vec{r}_{ic} + \vec{r}_c$,

$\sum_i\int\vec{F}_i\cdot d\vec{r}_{ic} + \int\vec{F}_{tot}\cdot d\vec{r}_c = \Delta\sum_i\frac{m_i}{2}(v_{ic}^2 + v_c^2 + 2v_{ic}\cdot v_c)$

$= \Delta\sum_i\frac{m_i}{2}v_{ic}^2 + \Delta\sum_i\frac{m_i}{2}v_c^2 + \Delta\sum m_i v_{ic}\cdot v_c$. $\sum m_i v_{ic} = 0$,

$\to \sum_i\int F_i\cdot dr_{ic} + \int F_{tot}\cdot dr_c = \Delta\sum_i\frac{m_i}{2}v_{ic}^2 + \Delta\frac{Mv_c^2}{2}$ *

$F_{tot} = \sum F_i = \sum m_i\dot{v}_i = M\dot{v}_c \to \int F_{tot}\cdot dr_c = M\int\frac{dv_c}{dt}\cdot dr_c = M\int dv_c\cdot v_c = \Delta\frac{Mv_c^2}{2}$

subtract from * and conclude.

~~Rope on a Table~~

~~rope of length l_0 slides from frictionless tabletop. Released from rest w/ 30 cm hanging over edge of table. Find time when left end of rope reaches end of table.~~

~~Tension does not do work to m since it's ⊥ to velocity~~

~~$\frac{mg^2}{}$ → v const. Thus,~~

~~$\omega l = \omega(l_0 - a\theta) = \omega_0 l_0$ or $\omega = \omega_0 l_0/(l_0 - a\theta)$. $N_z = \frac{dL_z}{dt}$~~

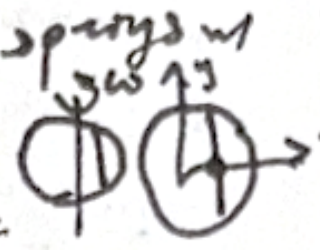
~~$N_z = -Ta$. $L_z = mlv = ml\omega_0 l_0 = m\omega_0 l_0(l_0 - a\theta)$~~

~~$\frac{dL_z}{dt} = -m\omega_0 l_0 a\omega = \frac{-ma\omega_0^2 l_0^2}{(l_0 - a\theta)}$~~

~~$-Ta = \frac{-ma\omega_0^2 l_0^2}{(l_0 - a\theta)}$ or $T = \frac{m\omega_0^2 l_0^2}{(l_0 - a\theta)}$~~

Mass in a Slot on a Disk

Disk rotating around vertical axis w/ ang. vel. ω. Mass m moving in a slot w/o friction attached to 2 identical springs w/ const k eq. pos. in middle.



In rot frame w/ disk force acting on m is

$\vec{F} = -2ky\hat{e}_y + m\omega\vec{r} + \vec{N} - 2m\omega\hat{e}_z\times\hat{e}_y$

$= -2ky\hat{e}_y + m\omega^2\vec{r} + \vec{N} + 2m\omega\hat{e}_x$

$F_x = 0$, $m\omega^2 x + N + 2m\omega = 0$

$F_y = m\ddot{y}$ $\quad -2ky + m\omega^2 y = m\ddot{y}$

$\to \ddot{y} + \left(\frac{2k}{m} - \omega^2\right)y = 0$. When $\frac{2k}{m} > \omega^2$, $y = 0$ is a stable point. Osc. freq. $\omega_0 = \sqrt{\frac{2k}{m} - \omega^2}$

Rope on a Table (see above)

Let x hang over table & L tot length of rope. $\frac{mgx}{L} = m\frac{dx^2}{dt^2}$

$\Rightarrow \ddot{x} = \frac{gx}{L}$. $x = Ae^{\omega t} + Be^{-\omega t}$

$\to \omega = \sqrt{\frac{g}{L}}$. from init. cond.

$A = B = \frac{x_0}{2}$, $x = x_0\cosh(\omega t)$

Falling Cube

edge length l (homogeneous) has one edge on a plane. Find angular. when face strikes plane.

(a) No sliding. Energy cons.

$\frac{mgl}{\sqrt{2}} = mg\frac{l}{2} + \frac{mv_{cm}^2}{2} + \frac{I\omega^2}{2}$ for v_{cm} is vel of CM when one face strikes plane. $v_{cm} = \frac{l}{\sqrt{2}}\omega$. $I = \frac{1}{6}ml^2$

$\to \frac{mgl}{2}(\sqrt{2} - 1) = \frac{1}{3}ml^2\omega^2$

$\to \omega^2 = \frac{3}{2}\frac{g}{l}(\sqrt{2} - 1)$

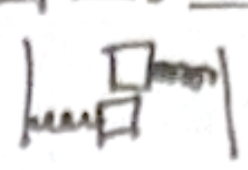
(b) Sliding. θ angle from vertical (b/w $0, \frac{\pi}{4}$).

$y = \frac{l}{\sqrt{2}}\cos\theta$ $\quad \dot{y}\Big|_{\theta=\frac{\pi}{4}} = -\frac{l}{\sqrt{2}}\sin\theta\dot{\theta}$

$= \frac{1}{2}l\dot{\theta} = -\frac{1}{2}l\omega^{\theta=\frac{\pi}{4}}$

cons. of energy gives $\omega^2 = \frac{12}{5}\frac{g}{l}(\sqrt{2} - 1)$.

Springs w/ Friction



Friction prop to rel inst. vel.

$F_{fr} = -\beta(\dot{x}_1 - \dot{x}_2)$

$\begin{cases} m\ddot{x}_1 + \beta(\dot{x}_1 - \dot{x}_2) + kx_1 = 0 \\ m\ddot{x}_2 + \beta(\dot{x}_1 + \dot{x}_2) + kx_2 = 0 \end{cases}$

Ansatz: $x_1(t) = B_1 e^{\alpha t}$, $x_2(t) = B_2 e^{\alpha t}$

where $\alpha = \lambda + i\omega$ is a quantity to be determined. Set det of coeffs of Bs to 0:

$(m\alpha^2 + \beta\alpha + k)^2 = \beta^2\alpha^2$

$\to \alpha_1 = \pm i\sqrt{\frac{k}{m}}$; $\omega_1 = \pm\sqrt{\frac{k}{m}}$

$\alpha_2 = \frac{1}{m}(-\beta \pm \sqrt{\beta^2 - mk})$

Bead on Rotating Hoop

Bead of mass m slides w/o friction on circular hoop of radius R. lies in a vertical plane & hoop rotates about vert. diam. w/ ang vel. Ω. Let θ be angle of bead from lowest pt on hoop.

Find eq. pos. & determine when bottom pt is unstable.

$r_\perp = R\sin\theta$ $\quad V_g = mgR(1 - \cos\theta)$

$V_c = -\frac{1}{2}m\Omega^2 r_\perp^2 = -\frac{1}{2}m\Omega^2 R^2\sin^2\theta$

$V_{eff} = V_g - V_c$. Eq @ $\frac{dV_{eff}}{d\theta} = 0$

$\to mR\sin\theta(g - \Omega^2 R\cos\theta) = 0$

$\to \sin\theta = 0$ or $\cos\theta = \frac{g}{\Omega^2 R}$

Stability near $\theta = 0$:

$V_{eff} = \frac{1}{2}mgR\theta^2 - \frac{1}{2}m\Omega^2 R^2\theta^2$

stable if $g - \Omega^2 R > 0$

Particle Moving Radially on a Rotating Disk

Bead slides outward along frictional horizontal radial slot on horizontal disk rotating w/ ang. sp. Ω.

In rot. fr. of disk, bead moves outward w/ sp. $\dot{r}$. Find Coriolis force & sideways force exerted by slot on bead.

$V_{rot} = \dot{r}\hat{r}$ $\quad \Omega = \Omega\hat{z}$

$F_{cor} = -2\Omega\times V_{rot}$

$\hat{z}\times\hat{r} = \hat{\phi}$

$\to F_{cor} = -2m\Omega\dot{r}\hat{\phi}$

Can't accel sideways rel. to slot so $N_{slot} = -F_{cor}$

Rigid Body Swinging about a Pivot

Rigid body of mass M is pivoted about O. CoM is dist d from pivot. Mom. of Inertia about pivot is I_0. Find small osc. freq.

$\tau = -Mgd\sin\theta$, $I_0\ddot{\theta} = -Mgd\sin\theta$

$\Rightarrow \omega = \sqrt{\frac{Mgd}{I_0}}$. $T = 2\pi\sqrt{\frac{I_0}{Mgd}}$

Rolling w/ Slipping

$v_{cm} \neq \omega R$

object slides right w/ init speed V_0, no initial spin. $a = -\mu_k g$

$\tau = f_k R = I\alpha$ & ass. $I = kMR^2$.

$\to \alpha = \frac{\mu_k g}{kR}$ $\quad v(t) = v_0 - \mu_k g t$

$\omega(t) = \frac{\mu_k g}{kR}t$ $\quad$ Rolling begins when $v = \omega R$

$\to t = \frac{kv_0}{\mu_k g(k+1)}$

Rolling w/o Slipping

Solid cylinder of mass M & radius R rolls w/o slipping down incline of height h. Start from rest, find v at bottom. $I = \frac{1}{2}MR^2$

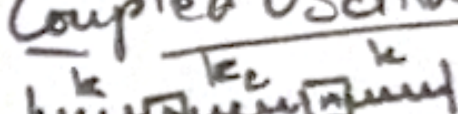
Cons. of Energy.

$Mgh = \frac{Mv^2}{2} + \frac{I\omega^2}{2}$

$= \frac{Mv^2}{2} + \frac{1}{4}Mv^2$

$\to v = \sqrt{\frac{4gh}{3}}$

Coupled Oscillators



init $x_1(0) = A$, $x_2(0) = 0$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$

EoM: $m\ddot{x}_1 = -kx_1 - k_c(x_1 - x_2)$

$m\ddot{x}_1 = -(k + k_c)x_1 + k_c x_2$

$m\ddot{x}_2 = -kx_2 - k_c(x_2 - x_1)$

$m\ddot{x}_2 = k_c x_1 - (k + k_c)x_2$

we $q_+ = x_1 + x_2$

$q_- = x_1 - x_2$

$\to x_1 = \frac{q_+ + q_-}{2}$

$x_2 = \frac{q_+ - q_-}{2}$

adding eqns:

$m\ddot{q}_+ = -kq_+$

$\to \omega_+ = \sqrt{\frac{k}{m}}$

subtracting eqns:

$m\ddot{q}_- = -(k + 2k_c)q_-$

$\to \omega_- = \sqrt{\frac{k + 2k_c}{m}}$

from init. cond.

$q_+(t) = A\cos(\omega_+ t)$

$q_-(t) = A\cos(\omega_- t)$

convert + trig. ids.

$x_1(t) = A\cos\left(\frac{\omega_+ + \omega_-}{2}t\right)\cos\left(\frac{\omega_- - \omega_+}{2}t\right)$

$x_2(t) = A\sin\left(\frac{\omega_+ + \omega_-}{2}t\right)\sin\left(\frac{\omega_- - \omega_+}{2}t\right)$

$A_1(t) = A\left|\cos\left(\frac{\omega_- - \omega_+}{2}t\right)\right|$

$A_2(t) = A\left|\sin\left(\frac{\omega_- - \omega_+}{2}t\right)\right|$

transfer when $A_1 = 0$, $A_2 = A$

$t_{transfer} = \frac{\pi}{\omega_- - \omega_+}$ $\quad t_{tot} = 2t_{transf}$